CMSC 35401: The Interplay of Learning and Game Theory
(Autumn 2022)

Introduction to Game Theory (I)

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Outline

- Games and its Basic Representation
- Nash Equilibrium and its Computation
- Other (More General) Classes of Games

(Recall) Example 1: Prisoner's Dilemma

- Two members A,B of a criminal gang are arrested
- They are questioned in two separate rooms
  - No communications between them

A \ B	B stays silent	B betrays
A stays silent	-1, -1	-3, 0
A betrays	0, -3	$\boxed{-2, -2}$

Q: How should each prisoner act?

- Both of them betray, though (-1,-1) is better for both

Example 2: Traffic Light Game

- Two cars heading to orthogonal directions

B

A

	STOP	GO
STOP	(-3, -2)	(-3, 0)
GO	(0, -2)	(-100, -100)

Q: what are the equilibrium statuses?

Answer: (STOP, GO) and (GO, STOP)

Example 3: Rock-Paper-Scissor

Player 2

Player 1

	Rock	Paper	Scissor
Rock	(0, 0)	(-1, 1)	(1, -1)
Paper	(1, -1)	(0, 0)	(-1, 1)
Scissor	(-1, 1)	(1, -1)	(0, 0)

Q: what is an equilibrium?

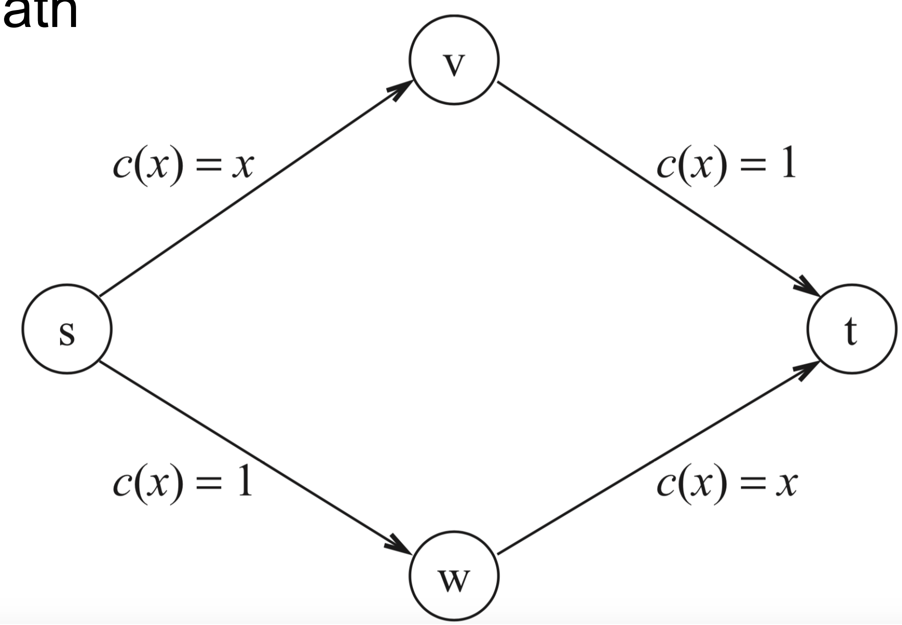
- Need to randomize – any deterministic action pair cannot make both players happy
- Common sense suggests (1/3,1/3,1/3)

Example 4: Selfish Routing

- One unit flow from s to t which consists of (infinite) individuals, each controlling an infinitesimal small amount of flow
- Each individual wants to minimize his own travel time

Q: What is the equilibrium status?

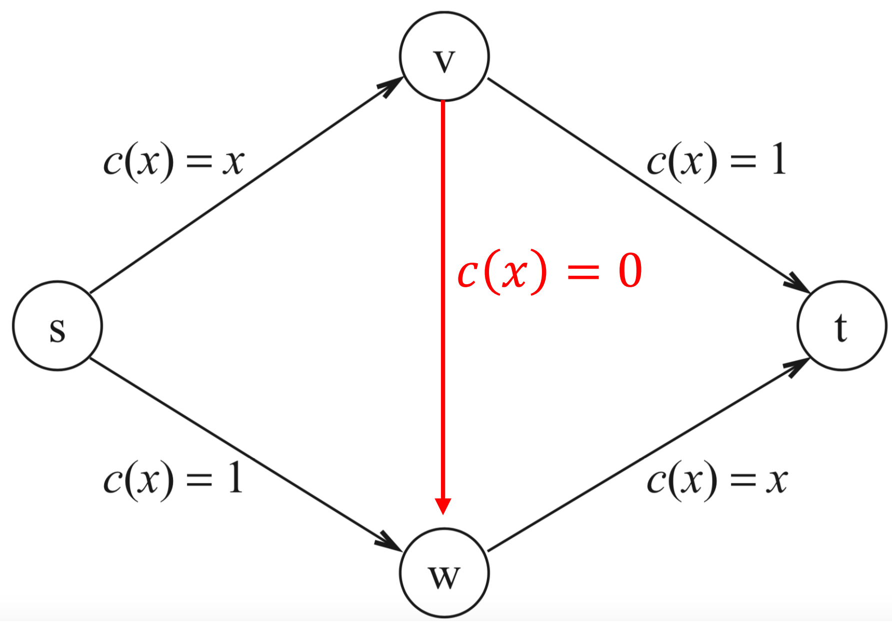
- Half unit flow through each path
- Social cost = 3/2



Example 4: Selfish Routing

- One unit flow from s to t which consists of (infinite) individuals, each controlling an infinitesimal small amount of flow
- Each individual wants to minimize his own travel time

Q: What is the equilibrium status after adding a superior high way with 0 traveling cost?

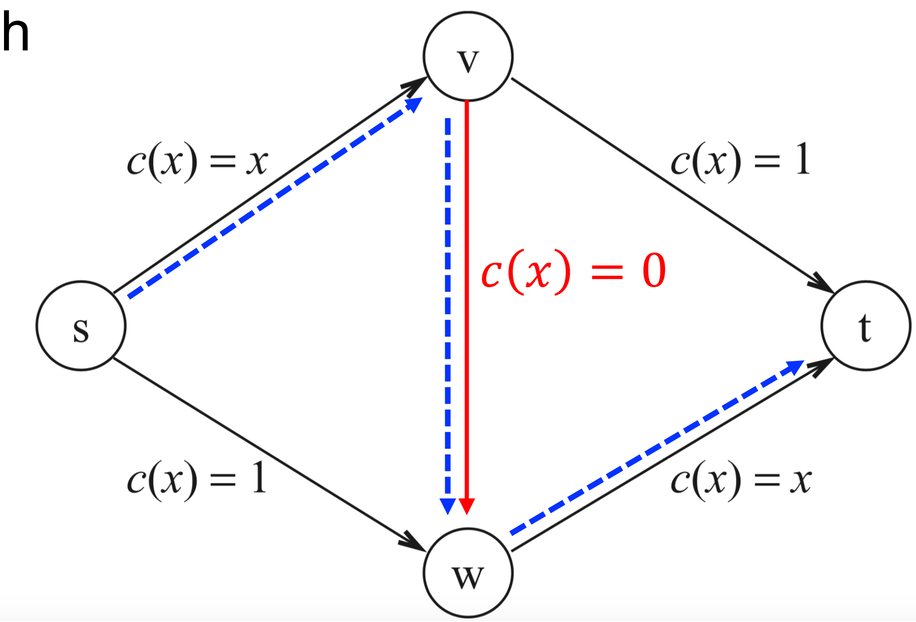


Example 4: Selfish Routing

- One unit flow from s to t which consists of (infinite) individuals, each controlling an infinitesimal small amount of flow
- Each individual wants to minimize his own travel time

Q: What is the equilibrium status after adding a superior high way with 0 traveling cost?

- Everyone takes the blue path
- Social cost = 2

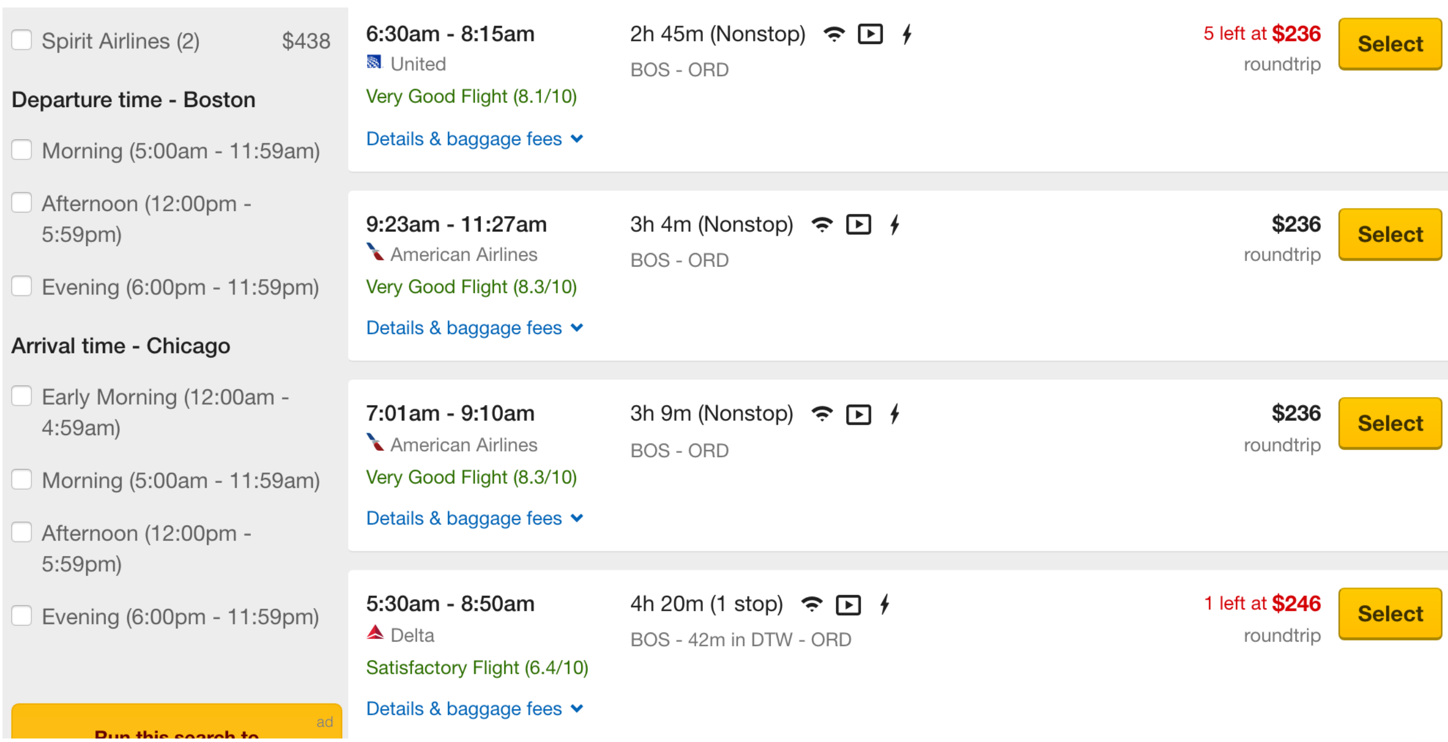


Key Characteristics of These Games

- Each agent wants to maximize her own payoff
- An agent’s payoff depends on other agents’ actions
- The interaction stabilizes at a state where no agent can increase his payoff via unilateral deviation

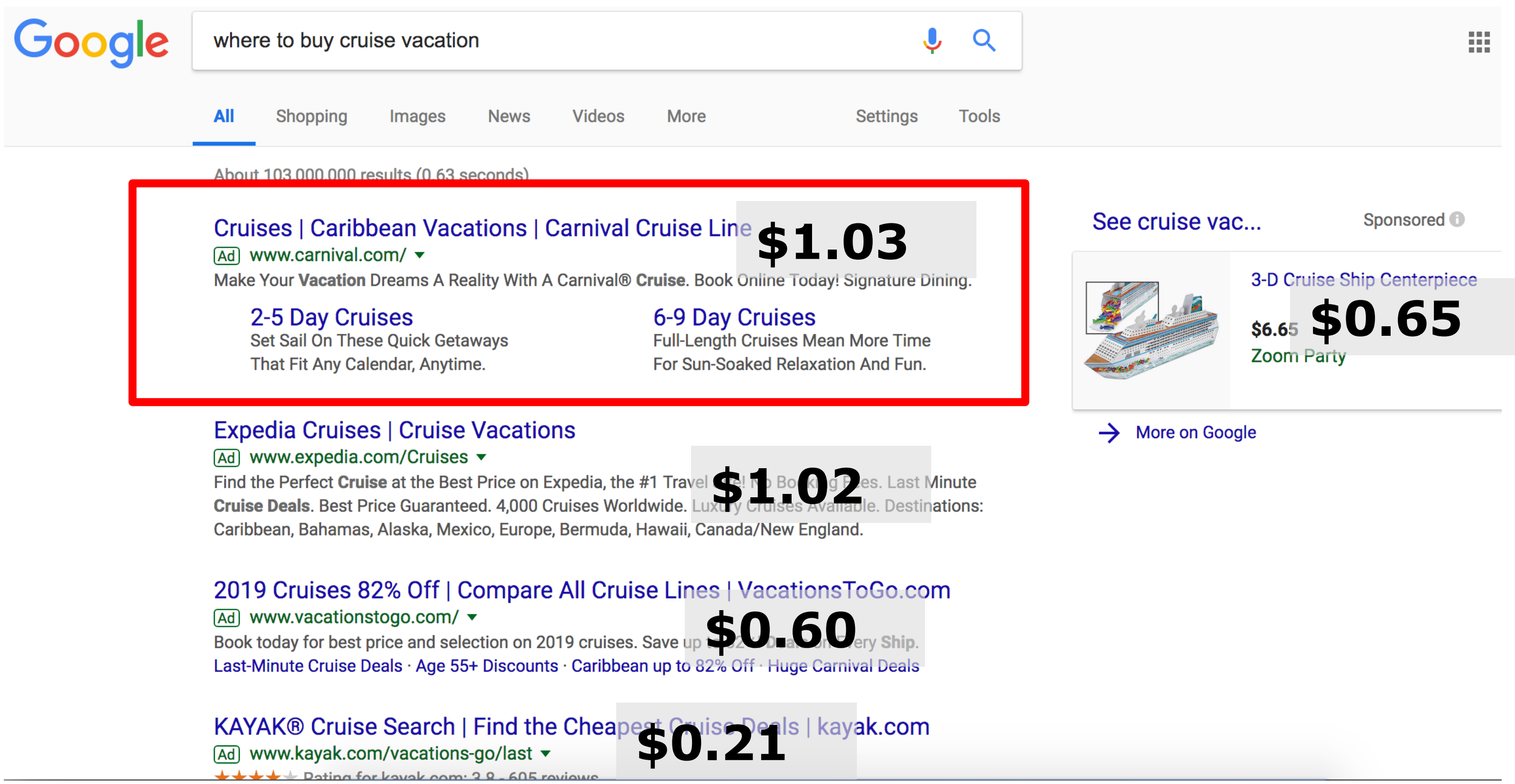
Strategic Games Are Ubiquitous

➢ Pricing



Strategic Games Are Ubiquitous

- Pricing
- Sponsored search
  - Drives 90%+ of Google's revenue



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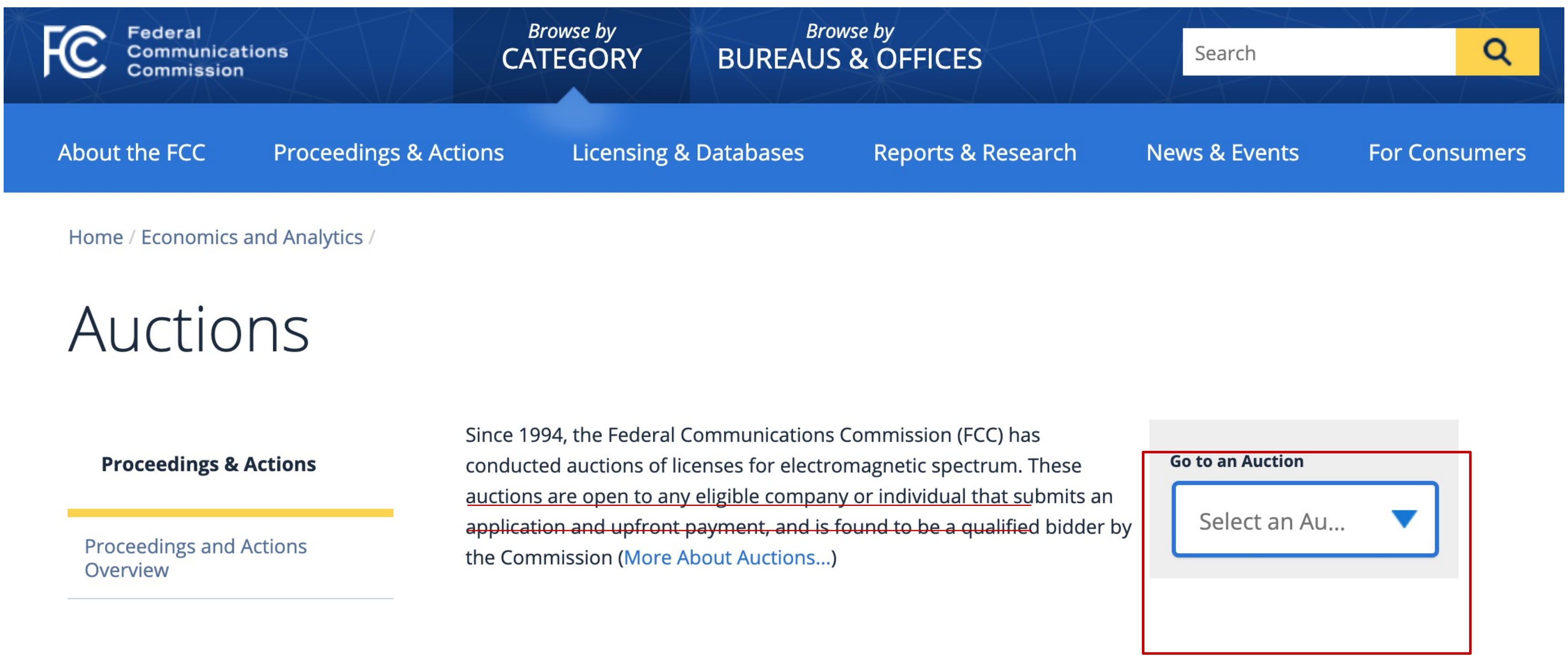
- FCC's Allocation of spectrum to radio frequency users



Strategic Games Are Ubiquitous

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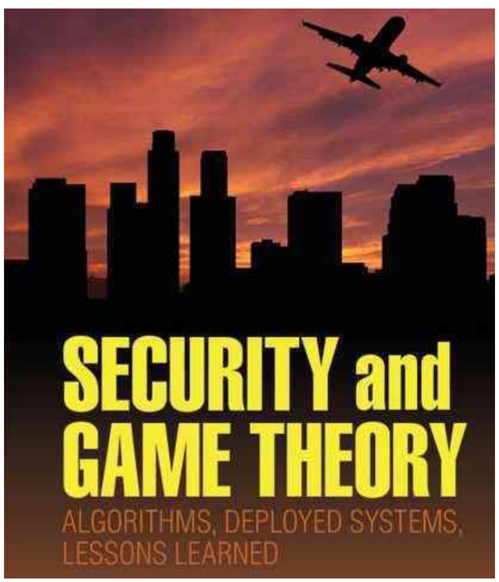
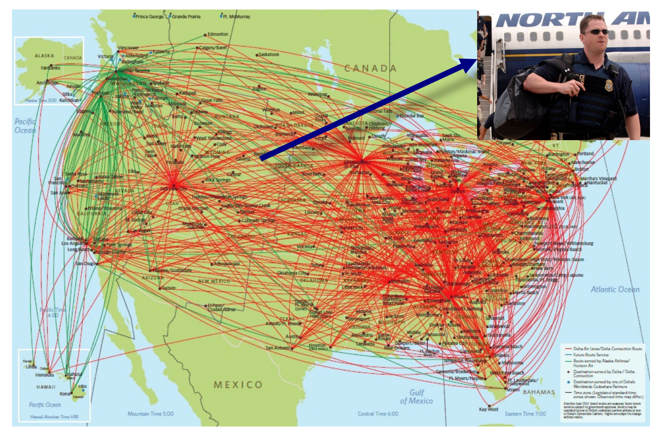
- ➢FCC's Allocation of spectrum to radio frequency users
- ➢National security, boarder patrolling, counter-terrorism





Optimize resource allocation against attackers/adversaries

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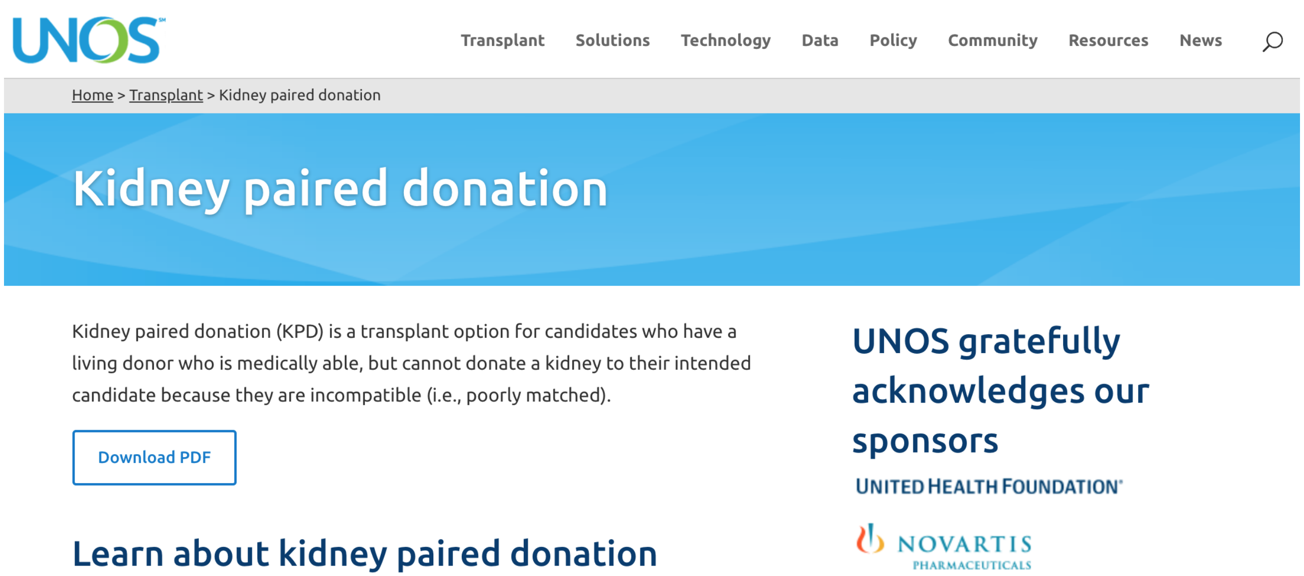
- National security, boarder patrolling, counter-terrorism
- Kidney exchange – decides who gets which kidney at when



Strategic Games Are Ubiquitous

- Pricing
- Sponsored search
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- FCC's Allocation of spectr
- National security, boarder
- Kidney exchange – decide

- Entertainment games: poker, blackjack, Go, chess . . .
- Social choice problems such as voting, fair division, etc.

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- National security, boarder patrolling, counter-terrorism
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- Entertainment games: poker, blackjack, Go, chess . . .
- Social choice problems such as voting, fair division, etc.

These are just a few example domains *where computer science has made significant impacts*; There are many others.

Main Components of a Game

- Players: participants of the game, each may be an individual, organization, a machine or an algorithm, etc.
- Strategies: actions available to each player
- Outcome: the profile of player strategies
- Payoffs: a function mapping an outcome to a utility for each player

Normal-Form Representation

- n players, denoted by set $[n] = \{1, \cdots, n\}$
- Player i takes action $a_i \in A_i$
- An outcome is the action profile $a = (a_1, \cdots, a_n)$
  - As a convention, $a_{-i} = (a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_n)$ denotes all actions excluding a_i
- Player i receives payoff $u_i(a)$ for any outcome $a \in \Pi_{i=1}^{n} A_i$
  - $u_i(a) = u_i(a_i, a_{-i})$ depends on other players' actions
- $\{A_i, u_i\}_{i \in [n]}$ are public knowledge

This is the most basic game model

- There are game models with richer and more intricate structures

Illustration: Prisoner's Dilemma

- 2 players: 1 and 2
- $A_i = \{\text{silent}, \ \text{betray}\}$ for $i = 1,2$
- An outcome can be, e.g., $a = (\text{silent}, \ \text{silent})$
- $u_1(a), u_2(a)$ are pre-defined, e.g., $u_1(\text{silent}, \ \text{silent}) = -1$
- The whole game is public knowledge; players take actions simultaneously
  - Equivalently, take actions without knowing the others' actions

Dominant Strategy

An action a_i is a **dominant strategy** for player i if a_i is better than any other action $a_i' \in A_i$, regardless what actions other players take.

Formally,

$$u_i(a_i, a_{-i}) \geq u_i(a_i', a_{-i}), \quad \forall a_i' \neq a_i \text{ and } \forall a_{-i}$$

Note: "strategy" is just another term for "action"

A \ B	B stays silent	B betrays
A stays silent	-1, -1	-3, 0
A betrays	0, -3	-2, -2

Prisoner's Dilemma

- *Betray* is a dominant strategy for both
- Dominant strategies do not always exist
  - For example, the traffic light game

	STOP	GO
STOP	(-3, -2)	(-3, 0)
GO	(0, -2)	(-100, -100)

Equilibrium

- An outcome a^* is an equilibrium if no player has incentive to deviate unilaterally. More formally,

$$u_i(a_i^*, a_{-i}^*) \geq u_i(a_i, a_{-i}^*), \qquad \forall a_i \in A_i$$

  - A special case of Nash Equilibrium, a.k.a., *pure strategy NE*

- If each player has a dominant strategy, they form an equilibrium

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- If each player has a dominant strategy, they form an equilibrium
- But, an equilibrium does not need to consist of dominant strategies

Quiz: find equilibrium

	L	M	R
U	4,3	5,1	6,2
M	2,1	8,4	3,6
D	3,0	9,6	2, 5

Equilibrium

- An outcome a^* is an equilibrium if no player has incentive to deviate unilaterally. More formally,

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- If each player has a dominant strategy, they form an equilibrium
- But, an equilibrium does not need to consist of dominant strategies

What about this?

	Rock	Paper	Scissor
Rock	(0, 0)	(-1, 1)	(1, -1)
Paper	(1, -1)	(0, 0)	(-1, 1)
Scissor	(-1, 1)	(1, -1)	(0, 0)

Pure strategy NE does not always exist…

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- Games and its Basic Representation
- Nash Equilibrium and its Computation
- Other (More General) Classes of Games

Pure vs Mixed Strategy

- Pure strategy: take an action deterministically
- Mixed strategy: can randomize over actions
  - Described by a distribution x_i where $x_i(a_i) =$ prob. of taking action a_i
  - $|A_i|$-dimensional simplex $\Delta_{A_i} := \{x_i: \sum_{a_i \in A_i} x_i(a_i) = 1, x_i(a_i) \geq 0\}$ contains all possible mixed strategies for player i
  - Players draw their own actions *independently*
- Given strategy profile $x = (x_1, \cdots, x_n)$, expected utility of i is

$$\sum_{a \in A} u_i(a) \cdot \Pi_{i \in [n]} x_i(a_i)$$

  - Often denoted as $u_i(x)$ or $u(x_i, x_{-i})$ or $u_i(x_1, \cdots, x_n)$
  - When x_i corresponds to some pure strategy a_i, we also write $u_i(a_i, x_{-i})$
  - Fix x_{-i}, $u_i(x_i, x_{-i})$ is linear in x_i

Best Responses

Fix any x_{-i}, x_i^* is called a best response to x_{-i} if

$$u_i(x_i^*, x_{-i}) \geq u_i(x_i, x_{-i}), \quad \forall\, x_i \in \Delta_{A_i}.$$

Claim. There always exists a pure best response

Proof: linear program "max $u_i(x_i, x_{-i})$ subject to $x_i \in \Delta_{A_i}$" has a vertex optimal solution

Remark: If x_i^* is a best response to x_{-i}, then any a_i in the support of x_i^* (i.e., $x_i^*(a_i) > 0$) must be equally good and are all "pure" best responses

Nash Equilibrium (NE)

A mixed strategy profile $x^* = (x_1^*, \cdots, x_n^*)$ is a **Nash equilibrium** if

$$u_i(x_i^*, x_{-i}^*) \geq u_i(x_i, x_{-i}^*), \qquad \forall\, x_i \in \Delta_{A_i}, \forall i \in [n].$$

That is, for any i, x_i^* is a best response to x_{-i}^*.

Remarks

- An equivalent condition: $u_i(x_i^*, x_{-i}^*) \geq u_i(a_i, x_{-i}^*), \forall\, a_i \in A_i, \forall i \in [n]$
  - Since there always exists a pure best response
- It is not clear yet that such a mixed strategy profile would exist
  - Recall that pure strategy Nash equilibrium may not exist

Nash Equilibrium (NE)

Theorem (Nash, 1951): Every finite game (i.e., finite players and actions) admits at least one mixed strategy Nash equilibrium.

- A foundational result in game-theory
- Example: rock-paper-scissor – what is a mixed strategy NE?
  - $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is a best response to $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$

		$1/3$	$1/3$	$1/3$
		Rock	Paper	Scissor
ExpU = 0	Rock	(0, 0)	(-1, 1)	(1, -1)
ExpU = 0	Paper	(1, -1)	(0, 0)	(-1, 1)
ExpU = 0	Scissor	(-1, 1)	(1, -1)	(0, 0)

Nash Equilibrium (NE)

Theorem (Nash, 1951): Every finite game (i.e., finite players and actions) admits at least one mixed strategy Nash equilibrium.

- An equilibrium outcome is not necessarily the best for players
  - Equilibrium only describes where the game stabilizes at
  - Many researches on understanding how self-interested behaviors reduces overall social welfare (recall the selfish routing game)
- A game may have many, even infinitely many, NEs
  - Which equilibrium you think it will stabilize at? → the issue of equilibrium selection

A \ B	B stays silent	B betrays
A stays silent	-1, -1	-3, 0
A betrays	0, -3	-2, -2

Computing a NE

Why we want to compute?

- Reason 1: want to predict what would happen when a multi-agent system stabilizes
  - E.g., if each self-driving car optimizes the time for its driver, would the road be efficient overall?
  - If each seller on Amazon tries to optimize their own revenue, what would be the ultimate price?
- Reason 2: want to figure out best action to take
  - Just like why we want to solve single-agent optimization problem
  - E.g., want to figure out best GO/Poker agent strategy

Intractability of Finding a NE

Theorem: Computing a Nash equilibrium for any two-player normal-form game is PPAD-hard.

Note: PPAD-hard problems are believed to not admit poly time algorithm

- A two player game can be described by $2mn$ numbers – $u_1(i,j)$ and $u_2(i,j)$ where $i \in [m]$ is player 1's action and $j \in [n]$ is player 2's.
- Theorem implies no $\text{poly}(mn)$ time algorithm to compute an NE for any input game
- Ok, so what can we hope?
  - If the game has good structures, maybe we can find an NE efficiently
  - For example, zero-sum ($u_1(i,j), +u_2(i,j) = 0$ for all i,j), some resource allocation games

An Exponential-Time Alg for Two-Player Nash

- What if we know the support of the NE: S_1, S_2 for player 1 and 2?
- The NE can be formulated by a linear feasibility problem with variables $x_1^*,\ x_2^*,\ U_1,\ U_2$

$$\forall\, j \in S_2: \qquad \textstyle\sum_{i\in S_1} u_2(i,j\,)x_1^*(i) = U_2$$
$$\forall\, j \notin S_2: \qquad \textstyle\sum_{i\in S_1} u_2(i,j\,)x_1^*(i) \le U_2$$

An Exponential-Time Alg for Two-Player Nash

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- The NE can be formulated by a linear feasibility problem with variables $x_1^*,\ x_2^*,\ U_1,\ U_2$

$$
\begin{array}{ll}
\forall\, j \in S_2: & \sum_{i \in S_1} u_2(i,j)x_1^*(i) = U_2 \\
\forall\, j \notin S_2: & \sum_{i \in S_1} u_2(i,j)x_1^*(i) \le U_2 \\
 & \sum_{i \in [m]} x_1^*(i) = 1 \\
\forall\, i \notin S_1: & x_1^*(i) = 0 \\
\forall\, i \in [m]: & x_1^*(i) \ge 0
\end{array}
$$

An Exponential-Time Alg for Two-Player Nash

- What if we know the support of the NE: S_1, S_2 for player 1 and 2?
- The NE can be formulated by a linear feasibility problem with variables $x_1^*,\ x_2^*,\ U_1,\ U_2$

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\forall\, i \notin S_1: & x_1^*(i) = 0 \\
\forall\, i \in [m]: & x_1^*(i) \ge 0
\end{array}
$$

Symmetric constraints for player 2

- The challenge of computing a NE is to find the correct supports
  - No general tricks, typically just try all possibilities
  - Some pre-processing may help, e.g., eliminating dominated actions
- This approach does not work for > 2 players games (why?)

An Example

	A	B	C
D	0, 0	-1, 1	1, -1
E	1, -1	0, 0	-1, -2
F	-1, 1	1, -1	0, -2

Step 1: pre-processing

- Column player never wants to play C

An Example

	A	B	C
D	0, 0	-1, 1	1, -1
E	1, -1	0, 0	-1, -2
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Step 1: pre-processing

- Column player never wants to play C

An Example

		q_1	q_2	0
		A	B	C
0	D	0, 0	-1, 1	1, -1
p_2	E	1, -1	0, 0	-1, -2
p_3	F	-1, 1	1, -1	0, -2

Step 2: Guess support and parameterize the equilibrium

$$u_1(E, A) \times q_1 + u_1(E, B) \times q_2 = u$$

$$u_1(F, A) \times q_1 + u_1(F, B) \times q_2 = u$$

Row player indifferent between {E, F}

$$u_1(D, A) \times q_1 + u_1(D, B) \times q_2 \leq u$$

Row player prefers {E, F} over {D}

… same for column player

Solve LP for p_2, p_3, q_1, q_2, u, v

An Example

		$1/3$	$2/3$	0
		A	B	C
0	D	0, 0	-1, 1	1, -1
$2/3$	E	1, -1	0, 0	-1, -2
$1/3$	F	-1, 1	1, -1	0, -2

Turns out our guess of support is correct

- If not, LP will be infeasible;
- In general, try all possibilities of support → Nash's theorem guarantees that one of LP systems must be feasible

Intractability of Finding "Best" NE

Theorem: It is NP-hard to compute the NE that maximizes the sum of players' utilities or any single player's utility even in two-player games.

- Proofs of these results for NEs are beyond the scope of this course

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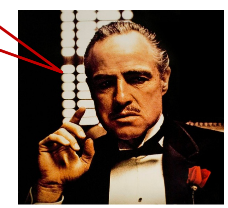
Bayesian Games

- Previously, assumed players have complete knowledge of the game
- What if players are uncertain about the game?
- Can be modeled as a Bayesian belief about the state of the game
  - This is typical in Bayesian decision making, but not the only way

B / A	B stays silent	B betrays
A stays silent	θ -1 -1 $+\theta$	0 -3 $+\theta$
A betrays	θ -3 0	-2 -2

I will give an additional reward θ for whoever staying silent



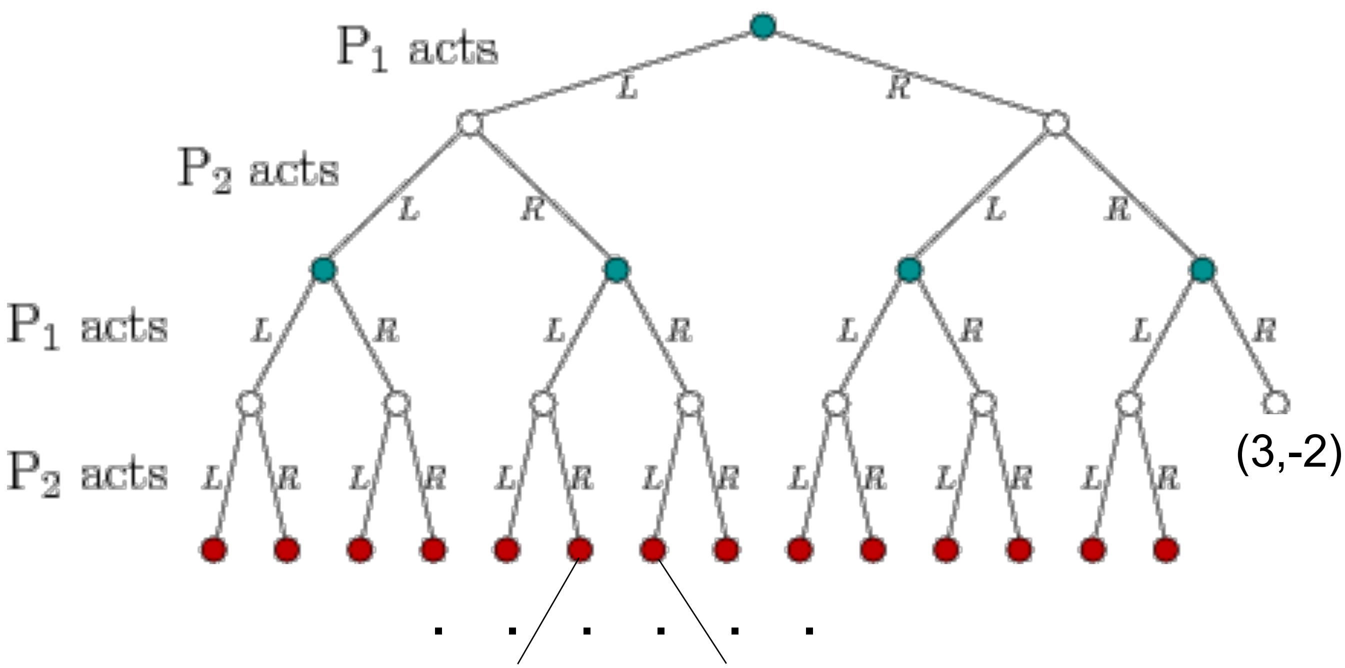
- It is believed that $\theta \in \{0,2,4\}$ uniformly at random
- Or maybe the two players have different beliefs about θ

Bayesian Games

- Previously, assumed players have complete knowledge of the game
- What if players are uncertain about the game?
- Can be modeled as a Bayesian belief about the state of the game
  - This is typical in Bayesian decision making, but not the only way
- More generally, can model player i' payoffs as u_i^θ where θ is a random state of the game
- Each player obtains a (random) signal s_i that is correlated with θ
  - A joint prior distribution over $(\theta, s_1, \cdots, s_n)$ is assumed the public knowledge
- Can define a similar notion as Nash equilibrium, but expected utility also incorporates the randomness of the state of the game θ
- Applications: poker, blackjack, auction design, etc.

Extensive-Form Games (EFGs)

- Previously, assumed players move only once and simultaneously
- More generally, can move sequentially and for multiple rounds
- Modeled by extensive-form game, described by a game tree



Extensive-Form Games (EFGs)

- Previously, assumed players move only once and simultaneously
- More generally, can move sequentially and for multiple rounds
- Modeled by extensive-form game, described by a game tree
- EFGs are extremely general, can represent almost all kinds of games, but of course very difficult to solve

A Remark

Sequential move fundamentally differs from simultaneous move

Nash equilibrium is only for simultaneous move

A Remark

Sequential move fundamentally differs from simultaneous move

Nash equilibrium is only for simultaneous move

- What is an NE?
  - (a_2, b_2) is the unique Nash, resulting in utility pair (1,2)
- If A moves first; B sees A's move and then best responds, how should A play?
  - Play action a_1 deterministically!

A \ B	b_1	b_2
a_1	(2, 1)	(-2, -2)
a_2	(2.01, -2)	(1, 2)

This sequential game model is called Stackelberg game, originally used to model market competition and now adversarial attacks.

Thank You

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