Announcements

➢ A related workshop at Northwestern this Friday

Link: https://www.ideal.northwestern.edu/events/elicitation-mechanisms-in-practice-workshop/

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Elicitation Mechanisms in Practice Workshop

Synopsis

Incentives for information procurement are integral to a wide range of applications including peer grading, peer review, prediction markets, crowd-sourcing, or conferring scientific credit. Meanwhile, mechanisms for information procurement have made large theoretical advances in recent years. This workshop will draw together practitioners that have deployed solutions in this space and experts in incentives and mechanisms to talk about existing connections, look for unexploited connections, and develop the next generation of information procurement research that will allow the theory to be further applied in these areas.

Speakers

Kevin Leyton-Brown (Univ. of British Columbia), Raul Castro Fernandez (UChicago), Yiling Chen (Harvard Univ.), and Nihar Shah (Carnegie Mellon Univ.).

CMSC 35401: The Interplay of Learning and Game Theory (Autumn 2022)

Introduction to Game Theory (II)

Instructor: Haifeng Xu

Outline

- Correlated and Coarse Correlated Equilibrium
- Zero-Sum Games
- GANs and Equilibrium Analysis

Recap: Normal-Form Games

- n players, denoted by set $[n] = \{1, \cdots, n\}$
- Player i takes action $a_i \in A_i$
- An outcome is the action profile $a = (a_1, \cdots, a_n)$
  - As a convention, $a_{-i} = (a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_n)$ denotes all actions excluding a_i
- Player i receives payoff $u_i(a)$ for any outcome $a \in \Pi_{i=1}^n A_i$
  - $u_i(a) = u_i(a_i, a_{-i})$ depends on other players' actions
- $\{A_i, u_i\}_{i \in [n]}$ are public knowledge

A mixed strategy profile $x^* = (x_1^*, \cdots, x_n^*)$ is a **Nash equilibrium (NE)** if for any i, x_i^* is a best response to x_{-i}^*.

NE Is Not the Only Solution Concept

- NE rests on two key assumptions
  1. Players move simultaneously (so they cannot see others' strategies before the move)

Sequential move fundamentally differs from simultaneous move

An Example

- What is an NE?
  - (a_2, b_2) is the unique Nash, resulting in utility pair (1,2)
- If A moves first; B sees A's move and then best responds, how should A play?
  - Play action a_1 deterministically!

B

A		b_1	b_2
	a_1	(2, 1)	(-2, -2)
	a_2	(2.01, -2)	(1, 2)

This sequential game model is called Stackelberg game, its equilibrium is called Strong Stackelberg equilibrium

An Example

When is sequential move more realistic?

- Market competition: market leader (e.g., Facebook) vs competing followers (e.g., small start-ups)
- Adversarial attacks: a learning algorithm vs an adversary, security agency vs real attackers
  - ✓ Used a lot in recent adversarial ML literature

This is precisely the reason that we need different equilibrium concepts to model different scenarios.

NE Is Not the Only Solution Concept

- NE rests on two key assumptions
  1. Players move simultaneously (so they cannot see others' strategies before the move)
  2. Players take actions independently

Today: we study what happens if players do not take actions independently but instead are "coordinated" by a central mediator

- This results in the study of correlated equilibrium

An Illustrative Example

B

A	STOP	GO
STOP	(-3, -2)	(-3, 0)
GO	(0, -2)	(-100, -100)

The Traffic Light Game

Well, we did not see many crushes in reality… Why?

- There is a mediator – the traffic light – that coordinates cars' moves
- For example, recommend (GO, STOP) for (A,B) with probability 3/5 and (STOP, GO) for (A,B) with probability 2/5
  - GO = green light, STOP = red light
  - Following the recommendation is a best response for each player
  - It turns out that this recommendation policy results in equal player utility $-6/5$ and thus is "fair"

This is how traffic lights are designed!

Correlated Equilibrium (CE)

- A (randomized) recommendation policy π assigns probability $\pi(a)$ for each action profile $a \in A = \Pi_{i \in [n]} A_i$
  - A mediator first samples $a \sim \pi$, then recommends a_i to i *privately*
- Upon receiving a recommendation a_i, player i's expected utility is

$$\frac{1}{c} \sum_{a_{-i} \in A_{-i}} u_i(a_i, a_{-i}) \cdot \pi(a_i, a_{-i})$$

  - c is a normalization term that equals the probability a_i is recommended

A recommendation policy π is a **correlated equilibrium** if

$$\sum_{a_{-i}} u_i(a_i, a_{-i}) \cdot \pi(a_i, a_{-i}) \geq \sum_{a_{-i}} u_i(a'_i, a_{-i}) \cdot \pi(a_i, a_{-i}), \forall a_i, a'_i \in A_i, \quad \forall i.$$

- That is, any recommended action to any player is a best response
  - CE makes incentive compatible action recommendations
- Assumed π is public knowledge so every player can calculate her utility

Basic Facts about Correlated Equilibrium

Fact. Any Nash equilibrium is also a correlated equilibrium.

- True by definition. Nash equilibrium can be viewed as independent action recommendation
- As a corollary, correlated equilibrium always exists

Fact. The set of correlated equilibria forms a convex set.

- In fact, distributions π satisfies a set of linear constraints

$$\sum_{a_{-i}} u_i(a_i, a_{-i}) \cdot \pi(a_i, a_{-i}) \geq \sum_{a_{-i}} u_i(a'_i, a_{-i}) \cdot \pi(a_i, a_{-i}), \forall a_i, a'_i \in A_i, \ \forall i \in [n]$$

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Fact. The set of correlated equilibria forms a convex set.

- In fact, distributions π satisfies a set of linear constraints
- This is nice because that allows us to optimize over all CEs
- Not true for Nash equilibrium

Coarse Correlated Equilibrium (CCE)

- A weaker notion of correlated equilibrium
- Also a recommendation policy π, but only requires that any player does not have incentives to opting out of our recommendations

A recommendation policy π is a **coarse correlated equilibrium** if

$$\sum_{a \in A} u_i(a) \cdot \pi(a) \geq \sum_{a \in A} u_i(a'_i, a_{-i}) \cdot \pi(a), \forall a'_i \in A_i, \forall i \in [n].$$

That is, for any player i, following π's recommendations is better than opting out of the recommendation and "acting on his own".

Compare to correlated equilibrium condition:

$$\sum_{a_{-i}} u_i(a_i, a_{-i}) \cdot \pi(a_i, a_{-i}) \geq \sum_{a_{-i}} u_i(a'_i, a_{-i}) \cdot \pi(a_i, a_{-i}), \forall a_i, a'_i \in A_i, \forall i.$$

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for any fixed a'_i

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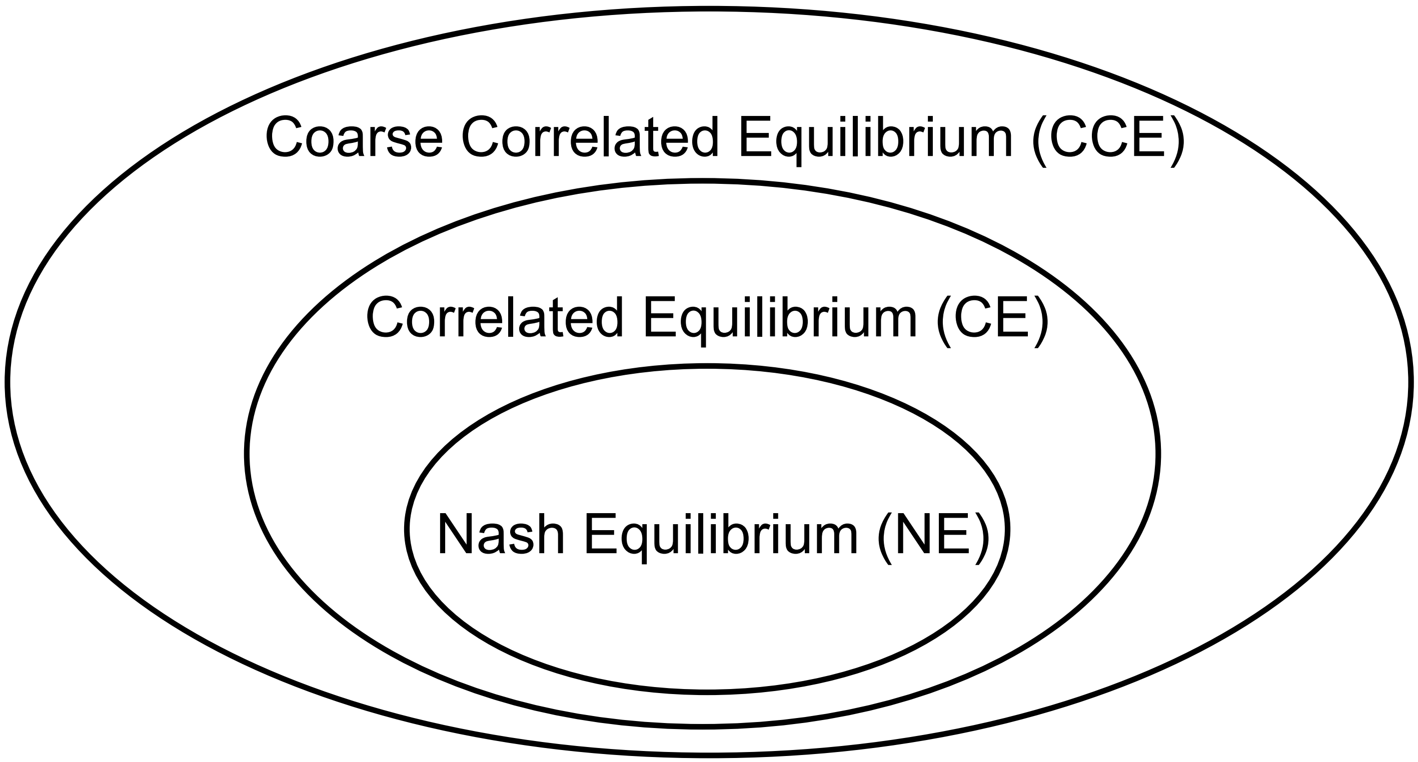
That is, for any player i, following π's recommendations is better than opting out of the recommendation and "acting on his own".

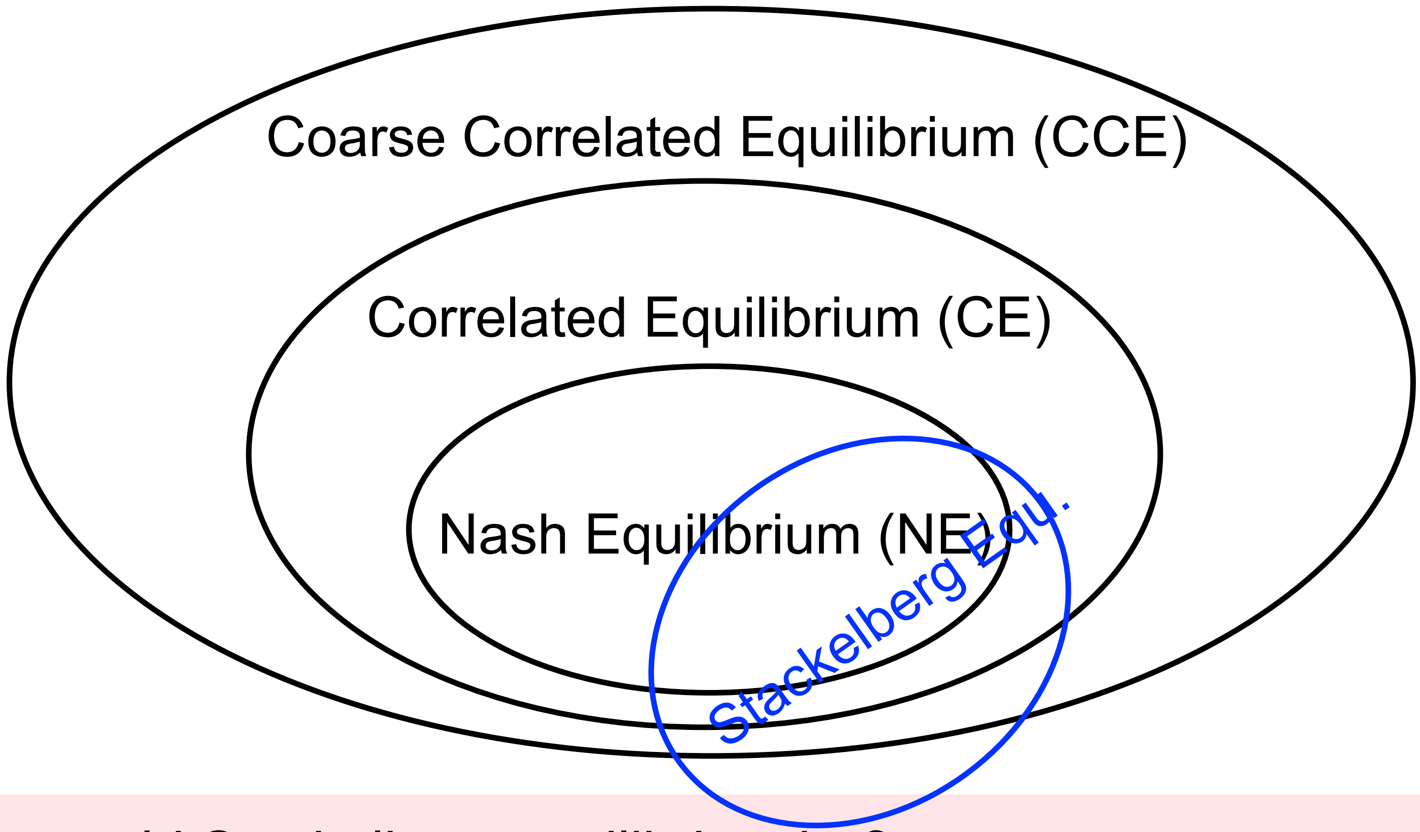
Fact. Any correlated equilibrium is a coarse correlated equilibrium.

The Equilibrium Hierarchy for Simultaneous-Move Games



There are other equilibrium concepts, but NE and CE are most often used. CCE is not used that often.

The Equilibrium Hierarchy for Simultaneous-Move Games



Where would Stackelberg equilibrium be?

- Not within any of them, somewhat different but also related
- See the paper titled "*On Stackelberg Mixed Strategies*" by Vincent Conitzer

Outline

- Correlated and Coarse Correlated Equilibrium
- Zero-Sum Games
- GANs and Equilibrium Analysis

Zero-Sum Games

- Two players: player 1 action $i \in [m] = \{1, \cdots, m\}$, player 2 action $j \in [n]$
- The game is **zero-sum** if $u_1(i,j) + u_2(i,j) = 0, \ \forall i \in [m], \ j \in [n]$
  - Models the strictly competitive scenarios
  - "Zero-sum" almost always mean "2-player zero-sum" games
  - n-player games can also be zero-sum, but not particularly interesting
- Let $u_1(x,y) = \sum_{i\in[m],j\in[n]} u_1(i,j)x_i y_j$ for any $x \in \Delta_m, y \in \Delta_n$
- (x^*, y^*) is a NE for the zero-sum game if: (1) $u_1(x^*, y^*) \ge u_1(i, y^*)$ for any $i \in [m]$; (2) $u_1(x^*, y^*) \le u_1(x^*, j)$ for any j $\in [m]$
  - Condition $u_1(x^*, y^*) \le u_1(x^*, j) \Leftrightarrow u_2(x^*, y^*) \ge u_2(x^*, j)$
  - We can "forget" u_2; Instead think of player 2 as minimizing player 1's utility

Maximin and Minimax Strategy

- Previous observations motivate the following definitions

Definition. $x^* \in \Delta_m$ is a maximin strategy of player 1 if it solves

$$\max_{x \in \Delta_m} \min_{j \in [n]} u_1(x, j).$$

The corresponding utility value is called maximin value of the game.

Remarks:

- x^* is player 1's best action if he was to move first

Maximin and Minimax Strategy

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$$\max_{x \in \Delta_m} \min_{j \in [n]} u_1(x, j).$$

The corresponding utility value is called maximin value of the game.

Definition. $y^* \in \Delta_n$ is a minimax strategy of player 2 if it solves

$$\min_{y \in \Delta_n} \max_{i \in [m]} u_1(i, y).$$

The corresponding utility value is called minimax value of the game.

Remark: y^* is player 2's best action if he was to move first

Duality of Maximin and Minimax

Fact. $\max_{x \in \Delta_m} \min_{j \in [n]} u_1(x, j) \le \min_{y \in \Delta_n} \max_{i \in [m]} u_1(i, y)$.

That is, moving first is no better in zero-sum games.

- Let $y^* = \arg\min_{y \in \Delta_n} \max_{i \in [m]} u_1(i, y)$, so

$$\min_{y \in \Delta_n} \max_{i \in [m]} u_1(i, y) = \max_{i \in [m]} u_1(i, y^*)$$

- We have

$$\max_{x \in \Delta_m} \min_{j \in [n]} u_1(x, j) \le \max_{x \in \Delta_m} u_1(x, y^*) = \max_{i \in [m]} u_1(i, y^*)$$

Duality of Maximin and Minimax

Fact. $\max_{x \in \Delta_m} \min_{j \in [n]} u_1(x, j) \le \min_{y \in \Delta_n} \max_{i \in [m]} u_1(i, y).$

Theorem. $\max_{x \in \Delta_m} \min_{j \in [n]} u_1(x, j) = \min_{y \in \Delta_n} \max_{i \in [m]} u_1(i, y).$

- Maximin and minimax can both be formulated as linear program

Maximin

$$
\begin{aligned}
\max \quad & u \\
\text{s.t.} \quad & u \le \textstyle\sum_{i=1}^{m} u_1(i, j)\, x_i, \quad \forall j \in [n] \\
& \textstyle\sum_{i=1}^{m} x_i = 1 \\
& x_i \ge 0, \quad \forall i \in [m]
\end{aligned}
$$

Minimax

$$
\begin{aligned}
\min \quad & v \\
\text{s.t.} \quad & v \ge \textstyle\sum_{j=1}^{n} u_1(i, j)\, y_j, \quad \forall i \in [m] \\
& \textstyle\sum_{j=1}^{n} y_j = 1 \\
& y_j \ge 0, \quad \forall j \in [n]
\end{aligned}
$$

- This turns out to be primal and dual LP. Strong duality yields the equation

"Uniqueness" of Nash Equilibrium (NE)

Theorem. In 2-player zero-sum games, (x^*, y^*) is a NE if and only if x^* and y^* are the maximin and minimax strategy, respectively.

$\Leftarrow$: if x^* [y^*] is the maximin [minimax] strategy, then (x^*, y^*) is a NE

- Want to prove $u_1(x^*, y^*) \geq u_1(i, y^*), \forall i \in [m]$

$$
\begin{aligned}
u_1(x^*, y^*) &\geq \min_{\mathrm{j}} u_1(x^*, j) \\
&= \max_{x \in \Delta_m} \min_{\mathrm{j}} u_1(x, j) \\
&= \min_{y \in \Delta_n} \max_{i \in [m]} u_1(i, y) \\
&= \max_{i \in [m]} u_1(i, y^*) \\
&\geq u_1(i, y^*), \forall i
\end{aligned}
$$

- Similar argument shows $u_1(x^*, y^*) \leq u_1(x^*, j), \forall j \in [n]$
- So (x^*, y^*) is a NE

“Uniqueness” of Nash Equilibrium (NE)

Theorem. In 2-player zero-sum games, (x^*, y^*) is a NE if and only if x^* and y^* are the maximin and minimax strategy, respectively.

⇒: if (x^*, y^*) is a NE, then x^* [y^*] is the maximin [minimax] strategy

- Observe the following inequalities

$$\begin{aligned} u_1(x^*, y^*) &= \max_{i \in [m]} u_1(i, y^*) \\ &\geq \min_{y \in \Delta_n} \max_{i \in [m]} u_1(i, y) \\ &= \max_{x \in \Delta_m} \min_{j} u_1(x, j) \\ &\geq \min_{j} u_1(x^*, j) \\ &= u_1(x^*, y^*) \end{aligned}$$

- So the two “≥” must both achieve equality.
  - The first equality implies y^* is the minimax strategy
  - The second equality implies x^* is the maximin strategy

“Uniqueness” of Nash Equilibrium (NE)

Theorem. In 2-player zero-sum games, (x^*, y^*) is a NE if and only if x^* and y^* are the maximin and minimax strategy, respectively.

Corollary.

- NE of any 2-player zero-sum game can be computed by LPs
- Players achieve the same utility in any Nash equilibrium.
  - Player 1’s NE utility always equals maximin (or minimax) value
  - This utility is also called the game value

The Collapse of Equilibrium Concepts in Zero-Sum Games

Theorem. In a 2-player zero-sum game, a player achieves the same utility in any Nash equilibrium, any correlated equilibrium, any coarse correlated equilibrium and any Strong Stackelberg equilibrium.

- Can be proved using similar proof techniques as for the previous theorem
- The problem of optimizing a player's utility over equilibrium can also be solved easily as the equilibrium utility is the same

Outline

- Correlated and Coarse Correlated Equilibrium
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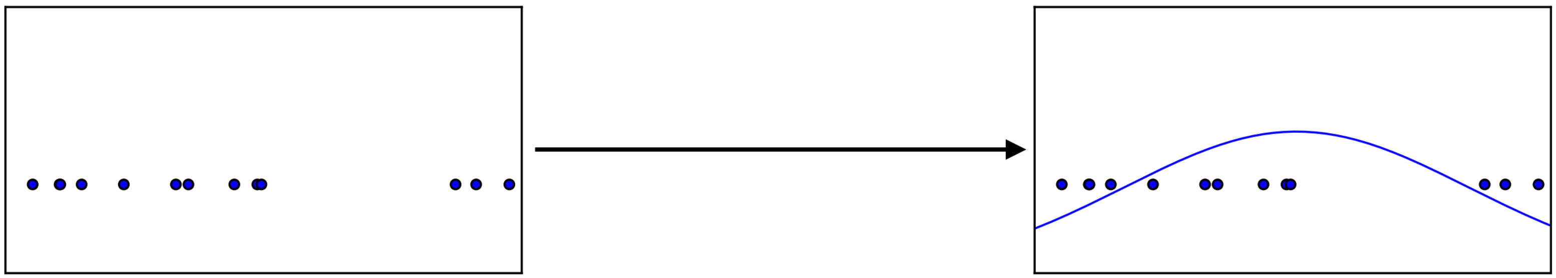
Generative Modeling



Input data points drawn from distribution P_{true}

Output data points drawn from distribution P_{model}

Goal: use data points from P_{true} to generate a P_{model} that is close to P_{true}

Applications

Celeb training data

[Karras et al. 2017]

Input images from
true distributions

Generated new images,
i.e., samples from P_{model}

A few another Demos:

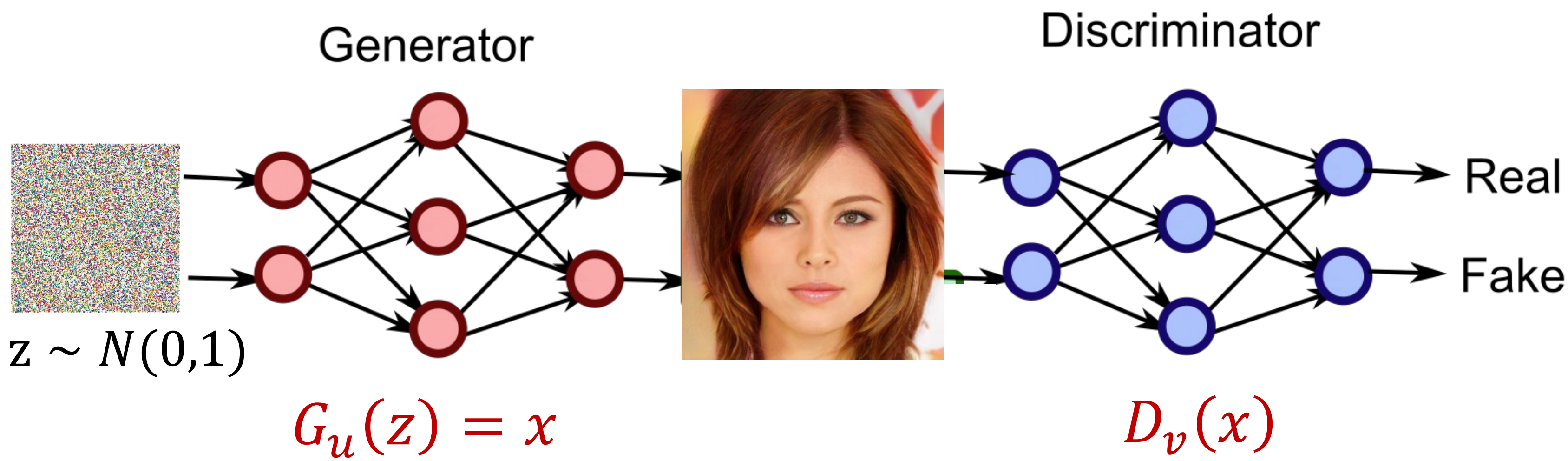
https://miro.medium.com/max/928/1*tUhgr3m54Qc80GU2BkaOiQ.gif

https://www.youtube.com/watch?v=PCBTZh41Ris&feature=youtu.be

http://ganpaint.io/demo/?project=church

GANs: Generative Adversarial Networks

- GAN is one particular generative model – a zero-sum game between the Generator and Discriminator



Objective: select model parameter u such that distribution of $G_u(z)$, denoted as P_{model}, is close to P_{real}

Objective: select model parameter v such that $D_v(x)$ is large if $x \sim P_{\text{real}}$ and $D_v(x)$ is small if $x \sim P_{\text{model}}$

GANs: Generative Adversarial Networks

- GAN is one particular generative model – a zero-sum game between the Generator and Discriminator
- The loss function originally formulated in [Goodfellow et al.'14]
  - $D_v(x) =$ probability of classifying x as "Real"
  - Log of the likelihood of being correct

$$L(u, v) = \mathbb{E}_{x \sim P_{\text{true}}} \log[D_v(x)] + \mathbb{E}_{z \sim N(0,1)} \log[1 - D_v(G_u(z))]$$

- The game: Discriminator maximizes this loss function whereas Generator minimizes this loss function
  - Results in the following zero-sum game

$$\min_u \max_v \; L(u, v)$$

  - The design of Discriminator is to improve training of Generator

GANs: Generative Adversarial Networks

- GAN is a large zero-sum game with intricate player payoffs
- Generator strategy G_u and Discriminator strategy D_v are typically deep neural networks, with parameters u, v
- Generator's utility function has the following general form where ϕ is an increasing concave function (e.g., $\phi(x) = \log x, x$ etc.)

$$\mathbb{E}_{x\sim P_{\text{true}}}\phi([D_v(x)]) + \mathbb{E}_{z\sim N(0,1)}\phi([1 - D_v(G_u(z))])$$

GAN research is essentially about modeling and solving this extremely large zero-sum game for various applications

WGAN – A Popular Variant of GAN

- Drawbacks of log-likelihood loss: unbounded at boundary, unstable
- Wasserstein GAN is a popular variant using a different loss function
  - I.e., substitute log-likelihood by the likelihood itself

$$\mathbb{E}_{x\sim P_{\mathrm{true}}} D_v(x) - \mathbb{E}_{z\sim N(0,1)} D_v(G_u(z))$$

  - Training is typically more stable

Research Challenges in GANs

$$\min_{u} \max_{v} \; \mathbb{E}_{x \sim P_{\text{true}}} \phi([D_v(x)]) + \mathbb{E}_{z \sim N(0,1)} \phi([1 - D_v(G_u(z))])$$

- What are the correct choice of loss function ϕ?
- What neural network structure for G_u and D_v?
- Only pure strategies allowed – equilibrium may not exist or is not unique due to non-convexity of strategies and loss function
- Do not know P_{true} exactly but only have samples
- How to optimize parameters u, v?
- . . .

A Basic Question

Even if we computed the equilibrium w.r.t. some loss function, does that really mean we generated a distribution close to P_{true}?

Research Challenges in GANs

$$\min_{u} \max_{v} \; \mathbb{E}_{x \sim P_{\text{true}}} \phi([D_v(x)]) + \mathbb{E}_{z \sim N(0,1)} \phi([1 - D_v(G_u(z))])$$

A Basic Question

Even if we computed the equilibrium w.r.t. some loss function, does that really mean we generated a distribution close to P_{true}?

- Intuitively, if the discriminator network D_v is strong enough, we should be able to get close to P_{true}
- Next, we will analyze the equilibrium of a stylized example

(Stylized) WGANs for Learning Mean

- True data drawn from $P_{\text{true}} = N(\alpha, 1)$
- Generator $G_u(z) = z + u$ where $z \sim N(0,1)$
- Discriminator $D_v(x) = vx$

Remarks:

a) Both Generator and Discriminator can be deep neural networks in general

b) We picked particular format for illustrative purpose and also convenience of theoretical analysis

(Stylized) WGANs for Learning Mean

- True data drawn from $P_{\text{true}} = N(\alpha, 1)$
- Generator $G_u(z) = z + u$ where $z \sim N(0,1)$
- Discriminator $D_v(x) = vx$
- WGAN then has the following close-form format

$$\min_u \max_v \ \mathbb{E}_{x \sim P_{\text{true}}}[D_v(x)] + \mathbb{E}_{z \sim N(0,1)}[1 - D_v(G_u(z))]$$

$$\Rightarrow \ \min_u \max_v \ \mathbb{E}_{x \sim N(\alpha,1)} \ [vx] \ + \mathbb{E}_{z \sim N(0,1)}[1 - v(z+u)]$$

$$\Rightarrow \ \min_u \max_v \ [v\alpha] \ + \ [1 - vu]$$

- This minimax problem solves to $u^* = \alpha$
- I.e, WGAN does precisely learn P_{true} at equilibrium in this case

See paper “**Generalization and Equilibrium in GANs**” by Arora et al. (2017) for more analysis regarding the equilibrium of GANs and whether they learn a good distribution at equilibrium

Thank You

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