

CMSC 3540I: The Interplay of Economics and ML
(Winter 2024)

Adversarial Multi-Armed Bandits

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Outline

- The Adversarial Multi-armed Bandit Problem
- A Basic Algorithm: Exp3
- Regret Analysis of Exp3

Recap: Online Learning So Far

Setup: T rounds; the following occurs at round t :

1. Learner picks a distribution p_t over actions $[n]$
2. Adversary picks cost vector $c_t \in [0,1]^n$
3. Action $i_t \sim p_t$ is chosen and learner incurs cost $c_t(i_t)$
4. Learner observes c_t (for use in future time steps)

Performance is typically measured by **regret**:

$$R_T = \sum_{i \in [n]} \sum_{t \in [T]} c_t(i) p_t(i) - \min_{j \in [n]} \sum_{t \in [T]} c_t(j)$$

The multiplicative weight update algorithm has regret $O(\sqrt{T \ln n})$.

Recap: Online Learning So Far

Convergence to equilibrium

- In repeated zero-sum games, if both players use a no-regret learning algorithm, their average strategy converges to an NE
- In general games, the average strategy converges to a CCE

Swap regret – a “stronger” regret concept and better convergence

- Def: each action i has a chance to deviate to another action $s(i)$
- In repeated general games, if both players use a no-swap-regret learning algorithm, their average strategy converges to a CE

There is a general reduction, converting any learning algorithm with regret R to one with swap regret nR .

This Lecture: Learning with Partial Feedback

- In online learning, the whole cost vector c_t can be observed by the learner, despite she only takes a single action i_t
 - Realistic in some applications, e.g., stock investment
- In many cases, we only see the reward of the action we take
 - For example: slot machines, a.k.a., [multi-armed bandits](#)



Other Applications with Partial Feedback

- Online advertisement placement or web ranking
 - Action: ad placement or ranking of webs
 - Cannot see the feedback for untaken actions

The screenshot shows a Google search for "pirate pants". The search bar is at the top with the Google logo and a search button. Below the search bar are tabs for "Web", "Shopping", "Images", "Videos", "News", "More", and "Search tools". The search results are displayed below, showing "About 1,990,000 results (0.51 seconds)".

The search results are divided into two main sections: "Shop for pirate pants on Google" (Sponsored) and "Images for pirate pants".

Shop for pirate pants on Google (Sponsored):

- Men Pirate Pants at Amazon:** www.amazon.com/fashion, 4.4 ★★★★★ rating for amazon.com. Shop hundreds of favorite brands. Free Shipping on Qualified Orders.
- Pirate Print Pants:** www.loudmouthgolf.com/Pants, Fashion That Comes In Loud Colors. Choose Your Style. Order Now!
- Pirate Pants & Trousers:** www.tobeapirate.com/, Complete your Pirate Outfit with authentic-design Pirate Pants.
- Target™ - Pirate Pants Kids:** www.target.com/, 4.3 ★★★★★ rating for target.com. Free Shipping On All Orders \$25+. Shop Pirate Pants Kids at Target™. 2099 Skokie Valley Rd, Highland Park (847) 266-8022.
- Pirate Pants 75% off:** www.sale-fire.com/Pirate+Pants, Save on Pirate Pants. Order today with free shipping!

Images for pirate pants: A row of five small images showing different styles of pirate pants. Below the images is a "Report images" link.

More images for pirate pants: A link to view more images.

Organic Search Results:

- Dress Like A Pirate - Dresslikeapirate.com:** <https://dresslikeapirate.com/>. Wench Garb, Gypsy Jewels, Frock Coats, Velvet Vests, Pirate Shirts, Lace Jabots, Harem Pants, Pirate Boots, Bellydance Wear, Leather Belts, Bodices, Gypsy ... Dress Like a Pirate - Pirate Men - Pirate Wenches - All Women's
- Pirate Pants, Knee Breeches N Slops – Pirate Fashions:** piratefashions.com/collections/pirate-pants-knee-breeches-n-slops. We have many options for ye: 2 versions of the classic Knee Breeches fer pirates who want confort, Buccaneer Pants fer gentlemen of fortune, n' 2 versions of th.
- Pirate clothing, nirate shirts, Pirate Pants, Pirate Boots, and**

Other Applications with Partial Feedback

- Online advertisement placement or web ranking
 - Action: ad placement or ranking of webs
 - Cannot see the feedback for untaken actions
- Recommendation system:
 - Action = recommended option (e.g., a restaurant)
 - Do not know other options' feedback

The screenshot shows the Yelp website interface for Lexington, MA. At the top, there is a search bar with the text "Search for (e.g. taco, cheap dinner, Max's)" and a "Near (Address, City, State or Zip)" field containing "Lexington, MA 02420". The Yelp logo is on the left. Below the search bar is a navigation menu with links: "Welcome", "About Me", "Write a Review", "Find Reviews", "Find Friends", "Messaging", "Talk", "Events", and "Member Search". The main content area is titled "Yelp Lexington" and includes a promotional banner: "Yelp is the best way to find great local businesses" with a "Create Your Free Account" button. Below this is a section titled "The Best of Lexington" with a list of categories: Restaurants (5,575 reviewed), Nightlife (940 reviewed), Food (2,960 reviewed), Shopping (4,337 reviewed), Bars (684 reviewed), and American (New) (424 reviewed). The "Restaurants" section is expanded, showing a list of top-rated businesses. The first is "1. Royal India Bistro" with a 4.5-star rating and 61 reviews, featuring a photo of the restaurant and a review snippet from David O.: "I had my favorite chicken tikka masala and it was really...". The second is "2. Wagon Wheel Nursery and Farm Stand" with a 4.0-star rating and 29 reviews. On the right side, there is a "Review of the Day" section featuring a review by Sarah D. for "Beantown Taqueria" with a 5-star rating and a snippet of her review: "I have been putting off writing this because I always get the same thing and I feel like I should probably branch out, but, nah." Below this is a "Yelp on the Go" section with a "Get it for free now" button and an image of the Yelp app on a smartphone.

Other Applications with Partial Feedback

- Online advertisement placement or web ranking
 - Action: ad placement or ranking of webs
 - Cannot see the feedback for untaken actions
- Recommendation system:
 - Action = recommended option (e.g., a restaurant)
 - Do not know other options' feedback
- Clinical trials
 - Action = a treatment
 - Don't know what would happen for treatments not chosen
- Playing strategic games
 - Cannot observe opponents' strategies but only know the payoff of the taken action
 - E.g., Poker games, competition in markets

Adversarial Multi-Armed Bandits (MAB)

- Very much like online learning, except **partial feedback**
 - The name “bandit” is inspired by slot machines
- Model: at each time step $t = 1, \dots, T$; the following occurs in order
 1. Learner picks a distribution p_t over **arms** $[n]$
 2. Adversary picks cost vector $c_t \in [0,1]^n$
 3. **Arm** $i_t \sim p_t$ is chosen and learner incurs cost $c_t(i_t)$
 4. Learner **only observes** $c_t(i_t)$ (for use in future time steps)
- Though we cannot observe c_t , adversary still picks c_t **before** i_t is sampled

Q: since learner does not observe $c_t(i)$ for $i \neq i_t$, can adversary arbitrarily modify these $c_t(i)$'s **after** i_t has been selected?

No, because this makes c_t depends on sampled i_t which is not allowed

Outline

- The Adversarial Multi-armed Bandit Problem
- A Basic Algorithm: Exp3
- Regret Analysis of Exp3

Recall the algorithm for full information setting:

Parameter: ϵ

Initialize weight $w_1(i) = 1, \forall i = 1, \dots, n$

For $t = 1, \dots, T$

1. Let $W_t = \sum_{i \in [n]} w_t(i)$, pick arm i with probability $w_t(i)/W_t$
2. Observe cost vector $c_t \in [0,1]^n$
3. For all $i \in [n]$, update $w_{t+1}(i) = w_t(i) \cdot (1 - \epsilon c_t(i))$

- In this lecture we will use this **exponential-weight** variant, and prove its regret bound
- Also called *Exponential Weight Update (EWU)*

Recall $1 - \delta \approx e^{-\delta}$ for small δ

Recall the algorithm for full information setting:

Parameter: ϵ


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For $t = 1, \dots, T$

1. Let $W_t = \sum_{i \in [n]} w_t(i)$, pick arm i with probability $w_t(i)/W_t$
2. Observe cost vector $c_t \in [0,1]^n$
3. For all $i \in [n]$, update $w_{t+1}(i) = w_t(i) \cdot e^{-\epsilon \cdot c_t(i)}$

Basic idea of Exp3

- Want to use EWU, but do not know vector $c_t \rightarrow$ try to estimate c_t !
- Well, we really only have $c_t(i_t)$, what can we do?

Estimate $\bar{c}_t = (0, \dots, 0, c_t(i_t), 0, \dots, 0)^T$?  Too optimistic

Estimate $\bar{c}_t = \left(0, \dots, 0, \frac{c_t(i_t)}{p_t(i_t)}, 0, \dots, 0\right)^T$ 

Exp3: a Basic Algorithm for Adversarial MAB

Parameter: ϵ

Initialize weight $w_1(i) = 1, \forall i = 1, \dots, n$

For $t = 1, \dots, T$

1. Let $W_t = \sum_{i \in [n]} w_t(i)$, pick arm i with probability $w_t(i)/W_t$
2. Sample action i_t and observe cost $c_t(i_t) \in [0,1]$
3. For all $i \in [n]$, update $w_{t+1}(i) = w_t(i) \cdot e^{-\epsilon \cdot \bar{c}_t(i)}$ where $\bar{c}_t = (0, \dots, 0, c_t(i_t)/p_t(i_t), 0, \dots, 0)^T$.

- That is, weight is updated only for the pulled arm
 - Because we really don't know how good are other arms at t
 - But i_t is more heavily penalized now
 - Attention: $c_t(i_t)/p_t(i_t)$ may be extremely large if $p_t(i_t)$ is small
- Called **Exp3**: Exponential-weight algorithm for Exploration and Exploitation

A Closer Look at the Estimator \bar{c}_t

- \bar{c}_t is random – it depends on the randomly sampled $i_t \sim p_t$
- \bar{c}_t is an unbiased estimator of c_t , i.e., $\mathbb{E}_{i_t \sim p_t} \bar{c}_t = c_t$
 - Because given p_t , for any i we have

$$\begin{aligned}\mathbb{E}_{i_t \sim p_t} \bar{c}_t(i) &= \mathbb{P}(i_t = i) \cdot \frac{c_t(i)}{p_t(i)} + \mathbb{P}(i_t \neq i) \cdot 0 \\ &= p_t(i) \cdot \frac{c_t(i)}{p_t(i)} \\ &= c_t(i)\end{aligned}$$

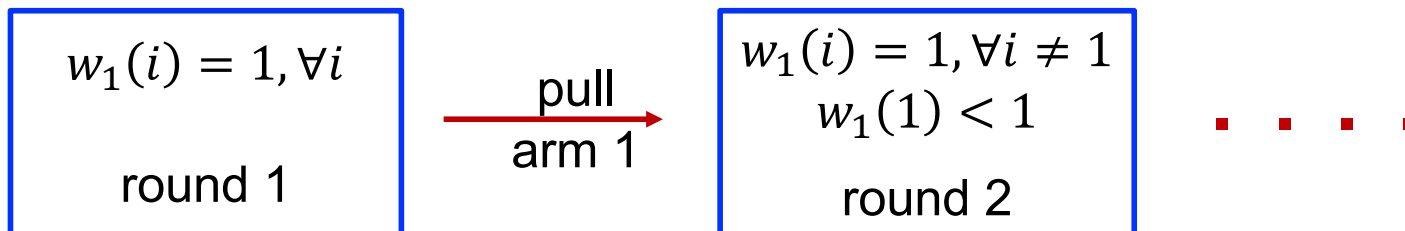
- This is exactly the reason for our choice of \bar{c}_t

Regret

$$R_T = \sum_{i \in [n]} \sum_{t \in [T]} c_t(i) p_t(i) - \min_{j \in [n]} \sum_{t \in [T]} c_t(j)$$

Key differences from full-feedback online learning

- R_T is random (even it already takes expectation over $i_t \sim p_t$)
 - Because distribution p_t itself is random, depends on sampled i_1, \dots, i_{t-1}
 - That is, if we run the same algorithm for multiple times, we will get different R_T value even when facing the same cost sequence!

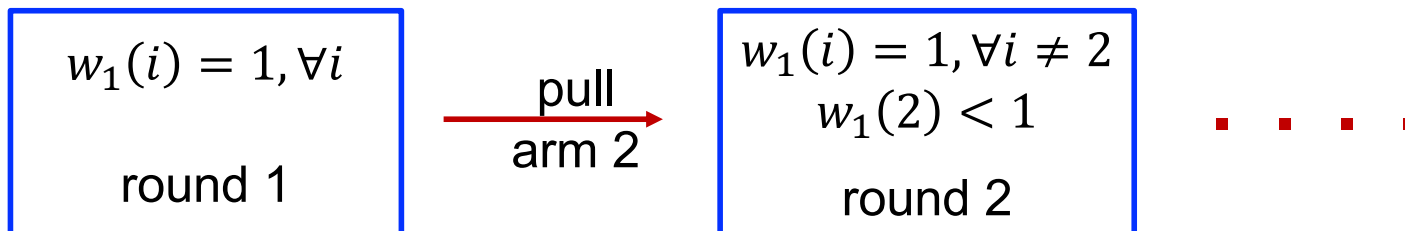


Regret

$$R_T = \sum_{i \in [n]} \sum_{t \in [T]} c_t(i) p_t(i) - \min_{j \in [n]} \sum_{t \in [T]} c_t(j)$$

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Regret

$$R_T = \sum_{i \in [n]} \sum_{t \in [T]} c_t(i) p_t(i) - \min_{j \in [n]} \sum_{t \in [T]} c_t(j)$$

Key differences from full-feedback online learning

- R_T is random (even it already takes expectation over $i_t \sim p_t$)
 - Because distribution p_t itself is random, depends on sampled i_1, \dots, i_{t-1}
 - That is, if we run the same algorithm for multiple times, we will get different R_T value even when facing the same cost sequence
- Cost vector c_t is also random as it generally depends on p_t
 - Adversary can observe p_t before coming up with cost vector c_t
- This is not the case in online learning
 - If we run the same algorithm for multiple times, we shall obtain the same R_T value *when facing the same adversary*

Regret

$$R_T = \sum_{i \in [n]} \sum_{t \in [T]} c_t(i) p_t(i) - \min_{j \in [n]} \sum_{t \in [T]} c_t(j)$$

- Therefore, in principle, we have to upper bound $\mathbb{E}(R_T)$ where expectation is over the **randomness of arm sampling**

$$\begin{aligned} \mathbb{E}(R_T) &= \mathbb{E} \left[\sum_{i \in [n]} \sum_{t \in [T]} c_t(i) p_t(i) - \min_{j \in [n]} \sum_{t \in [T]} c_t(j) \right] \\ &= \sum_{i \in [n]} \sum_{t \in [T]} \mathbb{E}[c_t(i) p_t(i)] - \mathbb{E} \left[\min_{j \in [n]} \sum_{t \in [T]} c_t(j) \right] \end{aligned}$$

by linearity of expectation

Regret

$$R_T = \sum_{i \in [n]} \sum_{t \in [T]} c_t(i) p_t(i) - \min_{j \in [n]} \sum_{t \in [T]} c_t(j)$$

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$$\begin{aligned} \mathbb{E}(R_T) &= \mathbb{E} \left[\sum_{i \in [n]} \sum_{t \in [T]} c_t(i) p_t(i) - \min_{j \in [n]} \sum_{t \in [T]} c_t(j) \right] \\ &= \sum_{i \in [n]} \sum_{t \in [T]} \mathbb{E}[c_t(i) p_t(i)] - \mathbb{E} \left[\min_{j \in [n]} \sum_{t \in [T]} c_t(j) \right] \\ &\geq \sum_{i \in [n]} \sum_{t \in [T]} \mathbb{E}[c_t(i) p_t(i)] - \min_{j \in [n]} \sum_{t \in [T]} \mathbb{E}[c_t(j)] \end{aligned}$$

because $\min_{j \in [n]} \sum_{t \in [T]} \mathbb{E}[c_t(j)] \geq \mathbb{E} \left[\min_{j \in [n]} \sum_{t \in [T]} c_t(j) \right]$

(proof: homework exercise)

Regret

$$R_T = \sum_{i \in [n]} \sum_{t \in [T]} c_t(i) p_t(i) - \min_{j \in [n]} \sum_{t \in [T]} c_t(j)$$

- Therefore, in principle, we have to upper bound $\mathbb{E}(R_T)$ where expectation is over the randomness of arm sampling

$$\begin{aligned} \mathbb{E}(R_T) &= \mathbb{E} \left[\sum_{i \in [n]} \sum_{t \in [T]} c_t(i) p_t(i) - \min_{j \in [n]} \sum_{t \in [T]} c_t(j) \right] \\ &= \sum_{i \in [n]} \sum_{t \in [T]} \mathbb{E}[c_t(i) p_t(i)] - \mathbb{E} \left[\min_{j \in [n]} \sum_{t \in [T]} c_t(j) \right] \\ &\geq \underbrace{\sum_{i \in [n]} \sum_{t \in [T]} \mathbb{E}[c_t(i) p_t(i)] - \min_{j \in [n]} \sum_{t \in [T]} \mathbb{E}[c_t(j)]}_{\text{Pseudo-Regret } \overline{R}_T} \end{aligned}$$

- Good regret guarantees good pseudo-regret, but not the reverse

Bounding regret turns out to be challenging

- Exp3 is not sufficient to guarantee small regret
- Next, we instead prove that Exp3 has small **pseudo-regret**
 - As is typical in many research papers
- A slight modification of Exp3 can be proved to have small regret

Outline

- The Adversarial Multi-armed Bandit Problem
- A Basic Algorithm: Exp3
- Regret Analysis of Exp3

Theorem. The pseudo regret of Exp3 is $O(\sqrt{nT \ln n})$.

High-level idea of the proof

- Pretend to be in the full information setting with cost equaling the estimated \bar{c}_t
- Relate \bar{c}_t to c_t since we know it is an unbiased estimator of c_t

Imitate a Full-Info Setting with Cost \bar{c}_t

- Recall regret bound for full information setting

$$R_T^{full} \leq \frac{\ln n}{\epsilon} + \epsilon T$$

- This holds for any cost vector, thus also \bar{c}_t
- But...one issue is that $\bar{c}_t(i_t)$ may be greater than 1
- Not a big issue – the same analysis yields the following bound

$$R_T^{full} \leq \frac{\ln n}{\epsilon} + \epsilon \max_i \sum_{t \in [T]} [\bar{c}_t(i)]^2$$

Real Issue: $\bar{c}_t(i)$ may be too large that we cannot bound R_T^{full}

Imitate a Full-Info Setting with Cost \bar{c}_t

A regret bound as follows turns out to work for our proof

$$R_T^{full} \leq \frac{\ln n}{\epsilon} + \epsilon \sum_t \sum_i p_t(i) [\bar{c}_t(i)]^2$$

- That is, instead of \max_i , the bound here averages over i
- This is clearly a better (i.e., smaller) regret upper bound
 - This turns out to be enough for us to bound the regret
 - Mathematically, the additional $p_t(i)$ term will help to cancel out a $p_t(i)$ denominator in $\bar{c}_t(i) = c_t(i)/p_t(i)$ term

Step 1: Tighter Regret for Full-Info Case

Lemma 1. The regret of the following algorithm is at most $\frac{\ln n}{\epsilon} + \frac{\epsilon}{2} \sum_t \sum_i p_t(i) [\bar{c}_t(i)]^2$ for any cost vector $\bar{c}_t \geq 0$.

Parameter: ϵ

Initialize weight $w_1(i) = 1, \forall i = 1, \dots, n$

For $t = 1, \dots, T$

1. Let $W_t = \sum_{i \in [n]} w_t(i)$, pick arm i with probability $w_t(i)/W_t$
2. Observe cost vector $\bar{c}_t \geq 0$
3. For all $i \in [n]$, update $w_{t+1}(i) = w_t(i) \cdot e^{-\epsilon \cdot \bar{c}_t(i)}$

Note: this yields a bound $\frac{\ln n}{\epsilon} + \frac{\epsilon}{2} T$ when $c_t \in [0,1]^n$

Step 1: Tighter Regret for Full-Info Case

Lemma 1. The regret of the following algorithm is at most $\frac{\ln n}{\epsilon} + \frac{\epsilon}{2} \sum_t \sum_i p_t(i) [\bar{c}_t(i)]^2$ for any cost vector $\bar{c}_t \geq 0$.

Proof: similar technique – carefully bound certain quantity

➤ Consider quantity $\sum_{i \in [n]} p_t(i) e^{-\epsilon \bar{c}_t(i)}$

Why this term?

- It tracks weight decrease (will be clear in next slide)
- The algebraic reasons, $e^{-\delta} \approx 1 - \delta + \delta^2/2$, which will give rise to both the term $p_t(i) \bar{c}_t(i)$ and $p_t(i) [\bar{c}_t(i)]^2$

Step 1: Tighter Regret for Full-Info Case

Lemma 1. The regret of the following algorithm is at most $\frac{\ln n}{\epsilon} + \frac{\epsilon}{2} \sum_t \sum_i p_t(i) [\bar{c}_t(i)]^2$ for any cost vector $\bar{c}_t \geq 0$.

➤ Consider quantity $\sum_{i \in [n]} p_t(i) e^{-\epsilon \bar{c}_t(i)}$

Fact 1. $\sum_{i \in [n]} p_t(i) e^{-\epsilon \bar{c}_t(i)} = W_{t+1} / W_t$, where $W_t = \sum_i w_t(i)$.

- The term $\sum_{i \in [n]} p_t(i) e^{-\epsilon \bar{c}_t(i)}$ is the decreasing rate of W_t
- Formal proof: HW exercise

Corollary. $\sum_t \log[\sum_{i \in [n]} p_t(i) e^{-\epsilon \bar{c}_t(i)}] = \log W_{T+1} - \log n$

- Telescope sum and $W_1 = n$

Step 1: Tighter Regret for Full-Info Case

Lemma 1. The regret of the following algorithm is at most $\frac{\ln n}{\epsilon} + \frac{\epsilon}{2} \sum_t \sum_i p_t(i) [\bar{c}_t(i)]^2$ for any cost vector $\bar{c}_t \geq 0$.

➤ Consider quantity $\sum_{i \in [n]} p_t(i) e^{-\epsilon \bar{c}_t(i)}$

Fact 2. $\sum_t \log[\sum_{i \in [n]} p_t(i) e^{-\epsilon \bar{c}_t(i)}] \leq -\epsilon \sum_{t,i} p_t(i) \bar{c}_t(i) + \frac{\epsilon^2}{2} \sum_{t,i} p_t(i) [\bar{c}_t(i)]^2$.

Follows from algebraic calculation

$$\sum_t \log[\sum_{i \in [n]} p_t(i) e^{-\epsilon \bar{c}_t(i)}] \leq \sum_t \log\left[\sum_{i \in [n]} p_t(i) \left[1 - \epsilon \bar{c}_t(i) + \frac{\epsilon^2}{2} [\bar{c}_t(i)]^2\right]\right]$$

$$\text{By } e^{-\delta} \leq 1 - \delta + \delta^2/2$$

Step 1: Tighter Regret for Full-Info Case

Lemma 1. The regret of the following algorithm is at most $\frac{\ln n}{\epsilon} + \frac{\epsilon}{2} \sum_t \sum_i p_t(i) [\bar{c}_t(i)]^2$ for any cost vector $\bar{c}_t \geq 0$.

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Follows from algebraic calculation

$$\begin{aligned} \sum_t \log[\sum_{i \in [n]} p_t(i) e^{-\epsilon \bar{c}_t(i)}] &\leq \sum_t \log \left[\sum_{i \in [n]} p_t(i) \left[1 - \epsilon \bar{c}_t(i) + \frac{\epsilon^2}{2} [\bar{c}_t(i)]^2 \right] \right] \\ &= \sum_t \log \left[1 - \sum_{i \in [n]} p_t(i) \epsilon \bar{c}_t(i) + \sum_{i \in [n]} p_t(i) \frac{\epsilon^2}{2} [\bar{c}_t(i)]^2 \right] \end{aligned}$$

Since $\sum_{i \in [n]} p_t(i) = 1$

Step 1: Tighter Regret for Full-Info Case

Lemma 1. The regret of the following algorithm is at most $\frac{\ln n}{\epsilon} + \frac{\epsilon}{2} \sum_t \sum_i p_t(i) [\bar{c}_t(i)]^2$ for any cost vector $\bar{c}_t \geq 0$.

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Follows from algebraic calculation

$$\begin{aligned} \sum_t \log[\sum_{i \in [n]} p_t(i) e^{-\epsilon \bar{c}_t(i)}] &\leq \sum_t \log\left[\sum_{i \in [n]} p_t(i) \left[1 - \epsilon \bar{c}_t(i) + \frac{\epsilon^2}{2} [\bar{c}_t(i)]^2\right]\right] \\ &= \sum_t \log\left[1 - \sum_{i \in [n]} p_t(i) \epsilon \bar{c}_t(i) + \sum_{i \in [n]} p_t(i) \frac{\epsilon^2}{2} [\bar{c}_t(i)]^2\right] \\ &\leq -\epsilon \sum_{t,i} p_t(i) \bar{c}_t(i) + \frac{\epsilon^2}{2} \sum_{t,i} p_t(i) [\bar{c}_t(i)]^2 \end{aligned}$$

Since $\log(1 + \delta) \leq \delta$ for any δ

Step 1: Tighter Regret for Full-Info Case

Lemma 1. The regret of the following algorithm is at most $\frac{\ln n}{\epsilon} + \frac{\epsilon}{2} \sum_t \sum_i p_t(i) [\bar{c}_t(i)]^2$ for any cost vector $\bar{c}_t \geq 0$.

- Consider quantity $\sum_{i \in [n]} p_t(i) e^{-\epsilon \bar{c}_t(i)}$
- Combining the two facts yields the lemma
 - HW exercise

Step 2: Relate \bar{c}_t to Pseudo-Regret

Lemma 2. $\sum_{t \in [T]} \mathbb{E}[c_t \cdot p_t - c_t(j)] = \sum_{t \in [T]} \mathbb{E}[\bar{c}_t \cdot p_t - \bar{c}_t(j)]$

- That is, expected pseudo regret from j w.r.t. true cost c_t equals that w.r.t. the estimated cost \bar{c}_t
(Both randomness come from EXP3's random action sample)

Recall pseudo-regret definition

$$\begin{aligned} \bar{R}_T &= \sum_{t \in [T]} \mathbb{E}[c_t \cdot p_t] - \min_{j \in [n]} \sum_{t \in [T]} \mathbb{E}[c_t(j)] \\ &= \max_{j \in [n]} \left[\sum_{t \in [T]} \mathbb{E}[c_t \cdot p_t] - \sum_{t \in [T]} \mathbb{E}[c_t(j)] \right] \\ &= \max_{j \in [n]} \underbrace{\sum_{t \in [T]} \mathbb{E}[c_t \cdot p_t - c_t(j)]}_{\text{Pseudo-regret from action } j} \end{aligned}$$

Step 2: Relate \bar{c}_t to Pseudo-Regret

Lemma 2. $\sum_{t \in [T]} \mathbb{E}[c_t \cdot p_t - c_t(j)] = \sum_{t \in [T]} \mathbb{E}[\bar{c}_t \cdot p_t - \bar{c}_t(j)]$

➤ Proof:

$$\mathbb{E}[\bar{c}_t \cdot p_t - \bar{c}_t(j)] = \mathbb{E}[\mathbb{E}[\bar{c}_t \cdot p_t - \bar{c}_t(j) | p_t]]$$

Because the randomness of \bar{c}_t comes:

1. Randomness of $i_t \sim p_t$
2. Randomness of p_t itself which depends on i_1, \dots, i_{t-1}

Step 2: Relate \bar{c}_t to Pseudo-Regret

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$$\begin{aligned} \mathbb{E}[\bar{c}_t \cdot p_t - \bar{c}_t(j)] &= \mathbb{E}[\mathbb{E}[\bar{c}_t \cdot p_t - \bar{c}_t(j) | p_t]] \\ &= \mathbb{E}[\mathbb{E}[c_t \cdot p_t - c_t(j) | p_t]] \end{aligned}$$

Because conditioning on p_t , \bar{c}_t is an unbiased estimator of c_t

Step 2: Relate \bar{c}_t to Pseudo-Regret

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Step 3: Derive Pseudo-Regret Bounds

Lemma 1. The regret of the following algorithm is at most $\frac{\ln n}{\epsilon} + \frac{\epsilon}{2} \sum_t \sum_i p_t(i) [\bar{c}_t(i)]^2$ for any cost vector $\bar{c}_t \geq 0$.

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➤ For any j , we have

$$\begin{aligned} \sum_{t \in [T]} \mathbb{E}[c_t \cdot p_t - c_t(j)] &= \mathbb{E}\left[\sum_{t \in [T]} [\bar{c}_t \cdot p_t - \bar{c}_t(j)]\right] \\ &\leq \mathbb{E}\left[\frac{\ln n}{\epsilon} + \frac{\epsilon}{2} \sum_t \sum_i p_t(i) [\bar{c}_t(i)]^2\right] \end{aligned}$$

By Lemma 1

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By conditional expectation

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By linearity of expectation

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$$\text{Observer } \mathbb{E}[[\bar{c}_t(i)]^2 | p_t] = 0 \cdot [1 - p_t(i)] + \left[\frac{c_t(i)}{p_t(i)} \right]^2 \cdot p_t(i) = \boxed{\frac{[c_t(i)]^2}{p_t(i)}}$$

Step 3: Derive Pseudo-Regret Bounds

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Pick $\epsilon = \sqrt{\frac{2 \ln n}{nT}}$ yields a regret bound of $O(\sqrt{nT \ln n})$

Summary of the Proof

- A tighter regret bound for full information setting
- Treat the (realized) estimated \bar{c}_t as the cost for full information
- Expected pseudo-regret w.r.t. to c_t equals expected pseudo-regret w.r.t. to \bar{c}_t
- Upper bound pseudo-regret by taking expectation over \bar{c}_t 's

The True Regret and Beyond

- Exp3 does not guarantee good true regret, still because $c_t(i)/p_t(i)$ may be too large
 - Pseudo-regret “smooths out” $p_t(i)$ by taking expectations first
- To obtain good true regret, need to modify Exp3 by adding some uniform exploration so that $p_t(i)$ is never too small
 - More intricate analysis, but gives the same regret bound $O(\sqrt{nT \ln n})$
- In addition to adversarial feedback, a “nicer” setting is when the cost of each arm is drawn from a **fixed but unknown** distribution
 - Called stochastic multi-armed bandits
 - Naturally, Exp3 and regret bound $O(\sqrt{nT \ln n})$ still applies
 - But a better algorithm called Upper-Confidence Bounds (UCB) yields much better regret bound $O(n \ln T)$
 - Different analysis techniques

Thank You

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