Announcements

- HW 1 will be out in these two days

CS6501: Topics in Learning and Game Theory
(Spring 2021)

Linear Programming Duality

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Slides of this lecture is adapted from https://www-bcf.usc.edu/~shaddin/cs675sp18/index.html

Outline

- Recap and Weak Duality
- Strong Duality and Its Proof
- Consequence of Strong Duality

Linear Program (LP)

General form:

$$
\begin{array}{lll}
\text{minimize (or maximize)} & c^T \cdot x & \\
\text{subject to} & a_i \cdot x \le b_i & \forall i \in C_1 \\
 & a_i \cdot x \ge b_i & \forall i \in C_2 \\
 & a_i \cdot x = b_i & \forall i \in C_3
\end{array}
$$

Standard form:

$$
\begin{array}{lll}
\text{maximize} & c^T \cdot x & \\
\text{subject to} & a_i \cdot x \le b_i & \forall i = 1, \cdots, m \\
 & x_j \ge 0 & \forall j = 1, \cdots, n
\end{array}
$$

Application: Optimal Production

- n products, m raw materials
- Every unit of product j uses a_{ij} units of raw material i
- There are b_i units of material i available
- Product j yields profit c_j per unit
- Factory wants to maximize profit subject to available raw materials

Can be formulated as an LP in *standard form*

$$\begin{array}{lll} \text{max} & c^T \cdot x & \\ \text{s.t.} & \sum_{j=1}^{n} a_{ij}\, x_j \le b_i, & \forall i \in [m] \\ & x_j \ge 0, & \forall j \in [n] \end{array}$$

Primal and Dual Linear Program

Primal LP

$$\begin{aligned} \max \quad & c^T \cdot x \\ \text{s.t.} \quad & \textstyle\sum_{j=1}^{n} a_{ij}\, x_j \le b_i, \quad && \forall i \in [m] \\ & x_j \ge 0, && \forall j \in [n] \end{aligned}$$

Dual LP

$$\begin{aligned} \min \quad & b^T \cdot y \\ \text{s.t.} \quad & \textstyle\sum_{i=1}^{m} a_{ij}\, y_i \ge c_j, \quad && \forall j \in [n] \\ & y_i \ge 0, && \forall i \in [m] \end{aligned}$$

Economic Interpretation:

Dual LP corresponds to the buyer's optimization problem, as follows:

- Buyer wants to directly buy the raw material
- Dual variable y_i is buyer's proposed price per unit of raw material i
- Dual price vector is feasible if factory is incentivized to sell materials
- Buyer wants to spend as little as possible to buy raw materials

Economic Interpretation

Primal LP

$$\max \quad c^T \cdot x$$

$$\text{s.t.} \quad \sum_{j=1}^{n} a_{ij}\, x_j \le b_i, \quad \forall i \in [m]$$

$$x_j \ge 0, \quad \forall j \in [n]$$

Dual LP

$$\min \quad b^T \cdot y$$

$$\text{s.t.} \quad \sum_{i=1}^{m} a_{ij}\, y_i \ge c_j, \quad \forall j \in [n]$$

$$y_i \ge 0, \quad \forall i \in [m]$$

price of material ←

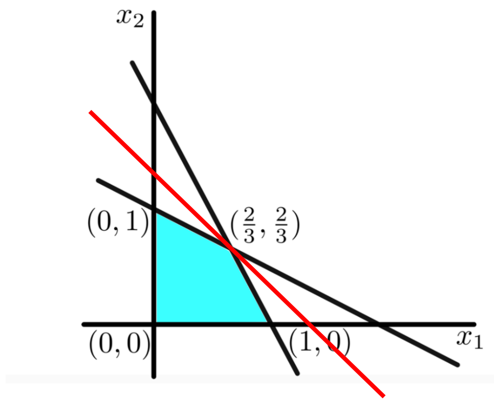
units of products

	x_1	x_2	x_3	x_4	
y_1	a_{11}	a_{12}	a_{13}	a_{14}	b_1
y_2	a_{21}	a_{22}	a_{23}	a_{24}	b_2
y_3	a_{31}	a_{32}	a_{33}	a_{34}	b_3
	c_1	c_2	c_3	c_4	

Interpretation II: Finding Best Upperbound

- Consider the simple LP from previous 2-D example

$$\begin{aligned} \text{maximize} \quad & x_1 + x_2 \\ \text{subject to} \quad & x_1 + 2x_2 \leq 2 \\ & 2x_1 + x_2 \leq 2 \\ & x_1, x_2 \geq 0 \end{aligned}$$



- We found that the optimal solution was at $(\frac{2}{3}, \frac{2}{3})$ with an optimal value of $\frac{4}{3}$.
- What if, instead of finding the optimal solution, we sought to find an upperbound on its value by combining inequalities?
  - Each inequality implies an upper bound of 2
  - Multiplying each by 1 and summing gives $x_1 + x_2 \leq 4/3$.

Interpretation II: Finding Best Upperbound

Primal LP

max $c^T \cdot x$

s.t. $Ax \leq b$

$x \geq 0$

Dual LP

min $b^T \cdot y$

s.t. $A^T y \geq c$

$y \geq 0$

- Multiplying each row i by y_i and summing gives the inequality

$$y^T A x \leq y^T b$$

(now we see why $y_i \geq 0$ when $a_i x \leq b_i$ but $y_i \in \mathbb{R}$ when $a_i x = b_i$)

- When $c^T \leq y^T A$, the right hand side of the inequality is an upper bound on $c^T x$ for every feasible x, because

$$c^T x \leq y^T A x \leq y^T b$$

- The dual LP can interpreted as finding the best upperbound on the primal that can be achieved this way.

Properties of Duals

- Duality is an inversion

Fact: Given any primal LP, the dual of its dual is itself.

Proof: homework exercise

Primal LP

$$
\begin{aligned}
\text{max} \quad & c^T \cdot x \\
\text{s.t.} \quad & \\
& a_i^T x \leq b_i, \quad \forall i \in C_1 \\
& a_i^T x = b_i, \quad \forall i \in C_2 \\
& x_j \geq 0, \quad \forall j \in D_1 \\
& x_j \in \mathbb{R}, \quad \forall j \in D_2
\end{aligned}
$$

Dual LP

$$
\begin{aligned}
\text{min} \quad & b^T \cdot y \\
\text{s.t.} \quad & \\
& \bar{a}_j y \geq c_j, \quad \forall j \in D_1 \\
& \bar{a}_j\, y = c_j, \quad \forall j \in D_2 \\
& y_i \geq 0, \quad \forall i \in C_1 \\
& y_i \in \mathbb{R}, \quad \forall i \in C_2
\end{aligned}
$$

- So far, mainly writing the Dual based on syntactic rules
- Next, will show Primal and Dual are inherently related

Weak Duality

Primal LP

$$\begin{aligned} \max \quad & c^t \cdot x \\ \text{s.t.} \quad & Ax \le b \\ & x \ge 0 \end{aligned}$$

Dual LP

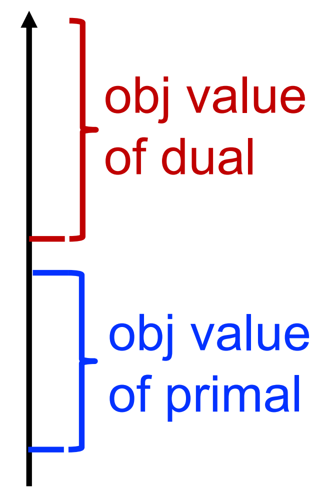
$$\begin{aligned} \min \quad & b^t \cdot y \\ \text{s.t.} \quad & A^t y \ge c \\ & y \ge 0 \end{aligned}$$

Theorem [Weak Duality]: For any primal feasible x and dual feasible y, we have $c^T \cdot x \le b^T \cdot y$

Corollary:

- If primal is unbounded, dual is infeasible
- If dual is unbounded, primal is infeasible
- If primal and dual are both feasible, then

  OPT(primal) ≤ OPT(dual)



Weak Duality

Primal LP

max $\quad c^t \cdot x$

s.t. $\quad Ax \leq b$

$\quad\quad x \geq 0$

Dual LP

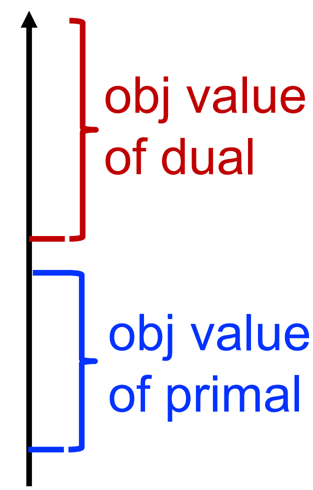
min $\quad b^t \cdot y$

s.t. $\quad A^t y \geq c$

$\quad\quad y \geq 0$

Theorem [Weak Duality]: For any primal feasible x and dual feasible y, we have $c^T \cdot x \leq b^T \cdot y$

Corollary: If x is primal feasible and y is dual feasible, and $c^T \cdot x = b^T \cdot y$, then both are optimal.



Interpretation of Weak Duality

Economic Interpretation:

If prices of raw materials are set such that there is incentive to sell raw materials directly, then factory's total revenue from sale of raw materials would exceed its profit from any production.

Upperbound Interpretation:

The method of rescaling and summing rows of the Primal indeed givens an upper bound of the Primal's objective value (well, self-evident…).

Proof of Weak Duality

Primal LP

max	$c^t \cdot x$
s.t.	$Ax \leq b$
	$x \geq 0$

Dual LP

min	$b^t \cdot y$
s.t.	$A^t y \geq c$
	$y \geq 0$

$$y^T \cdot b \geq y^T \cdot Ax = x^T \cdot A^T y \geq x^T \cdot c$$

Outline

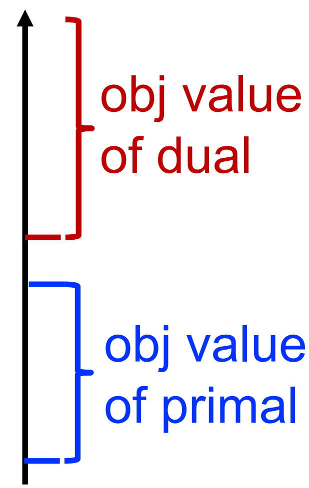
- Recap and Weak Duality
- Strong Duality and Its Proof
- Consequence of Strong Duality

Strong Duality

Theorem [Strong Duality]: If either the primal or dual is feasible and bounded, then so is the other and OPT(primal) = OPT(dual).

… I thought there was nothing worth publishing until the Minimax Theorem was proved.

John von Neumann



Interpretation of Strong Duality

Economic Interpretation:

There exist raw material prices such that the factory is indifferent between selling raw materials or products.

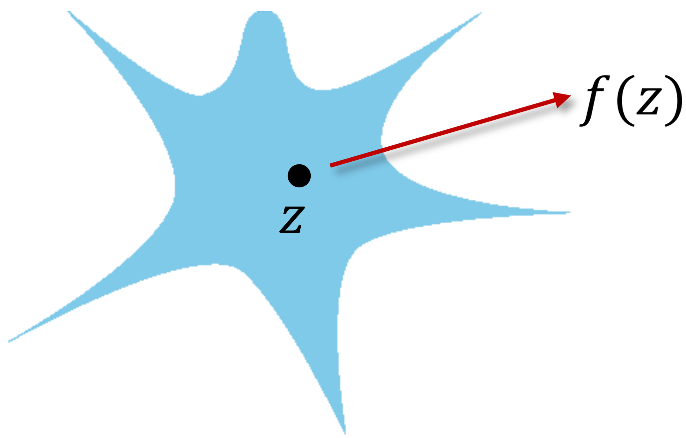
Upperbound Interpretation:

The method of scaling and summing constraints yields a tight upperbound for the primal objective value.

Proof of Strong Duality

Projection Lemma

Weierstrass' Theorem: Let Z be a compact set, and let $f(z)$ be a continuous function on z. Then $\min\{ f(z) : z \in Z \}$ exists.

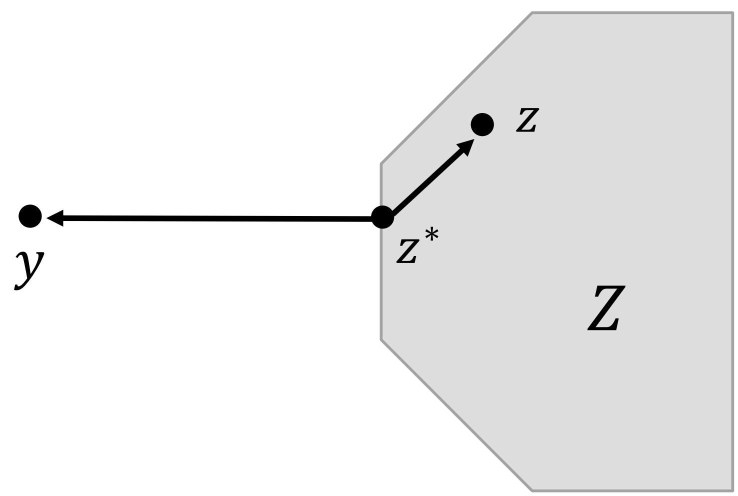


Projection Lemma

Weierstrass' Theorem: Let Z be a compact set, and let $f(z)$ be a continuous function on z. Then $\min\{ f(z) : z \in Z \}$ exists.

Projection Lemma: Let $Z \subset \mathbb{R}^m$ be a nonempty closed convex set and let $y \notin Z$. Then there exists $z^* \in Z$ with minimum l_2 distance from y. Moreover, $\forall z \in Z$ we have $(y - z^*)^T (z - z^*) \leq 0$.

Proof: homework exercise

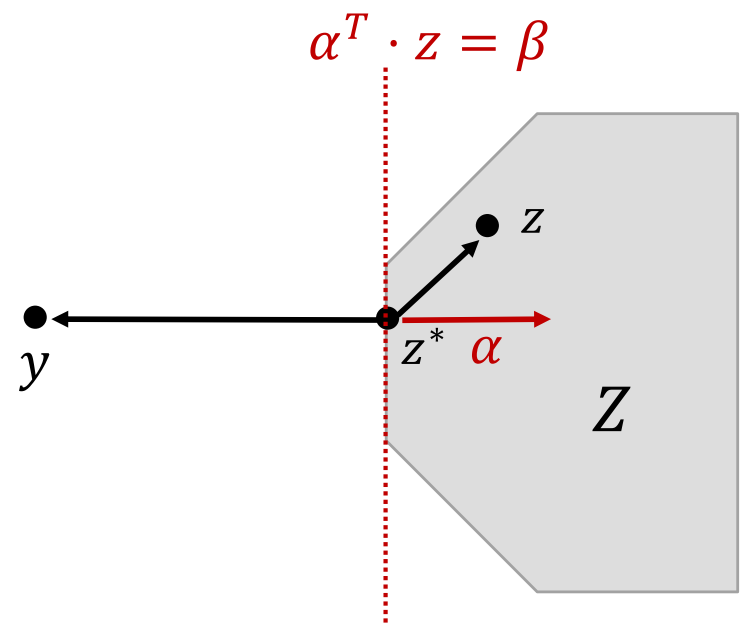


Separating Hyperplane Theorem

Theorem: Let $Z \subset \mathbb{R}^m$ be a nonempty closed convex set and let $y \notin Z$. Then there exists a hyperplane $\alpha^T \cdot z = \beta$ that strictly separates y from Z. That is, $\alpha^T \cdot z \geq \beta, \forall z \in Z$ and $\alpha^T \cdot y < \beta$.

Proof: choose $\alpha = z^* - y$ and $\beta = \alpha \cdot z^*$ and use projection lemma

- Homework exercise



Farkas' Lemma

Farkas' Lemma: Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$, then exactly one of the following two statements holds:

a) There exists $x \in \mathbb{R}^n$ such that $Ax = b$ and $x \geq 0$

b) There exists $y \in \mathbb{R}^m$ such that $A^T y \geq 0$ and $b^T y < 0$

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & a_{13} & a_{14} & b_1 \\ a_{21} & a_{22} & a_{23} & a_{24} & b_2 \\ a_{31} & a_{32} & a_{33} & a_{34} & b_3 \end{array}\right]$$

Farkas' Lemma

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b) There exists $y \in \mathbb{R}^m$ such that $A^T y \geq 0$ and $b^T y < 0$

Case b):

y_1	a_{11}	a_{12}	a_{13}	a_{14}	b_1
y_2	a_{21}	a_{22}	a_{23}	a_{24}	b_2
y_3	a_{31}	a_{32}	a_{33}	a_{34}	b_3

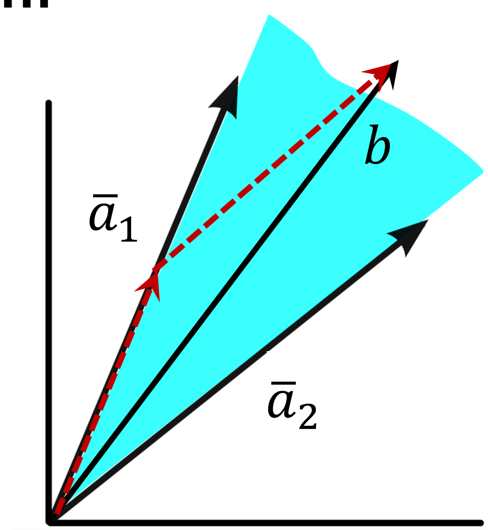
Farkas' Lemma

Farkas' Lemma: Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$, then exactly one of the following two statements holds:

a) There exists $x \in \mathbb{R}^n$ such that $Ax = b$ and $x \geq 0$

b) There exists $y \in \mathbb{R}^m$ such that $A^T y \geq 0$ and $b^T y < 0$

Geometric interpretation:



$\bar{a}_j$ is j'th column of A

a) b is in the cone

Farkas' Lemma

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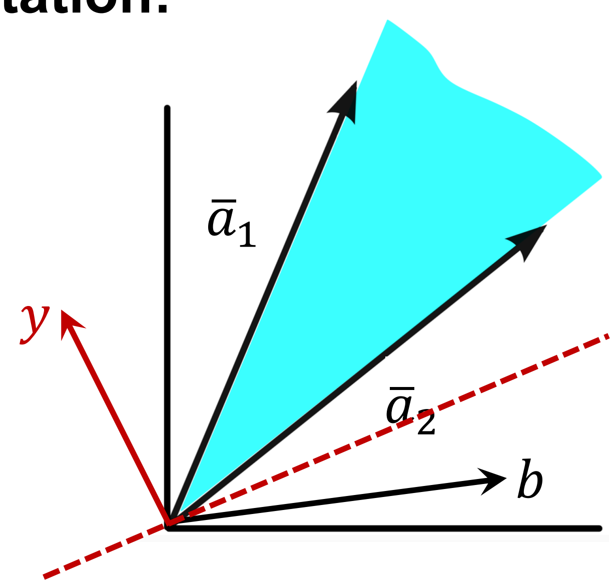
Geometric interpretation:



$\bar{a}_j$ is j'th column of A

a) b is in the cone

b) b is not in the cone, and there exists a hyperplane with direction y that separates b from the cone

Farkas' Lemma

Farkas' Lemma: Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$, then exactly one of the following two statements holds:

a) There exists $x \in \mathbb{R}^n$ such that $Ax = b$ and $x \geq 0$

b) There exists $y \in \mathbb{R}^m$ such that $A^T y \geq 0$ and $b^T y < 0$

Proof:

- Cannot both hold; Otherwise, yields contradiction as follows:

$$0 \leq (A^T y)^T \cdot x = y^T \cdot (Ax) = y^T \cdot b < 0.$$

- Next, we prove if (a) does not hold, then (b) must hold
  - This implies the lemma

Farkas' Lemma

Farkas' Lemma: Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$, then exactly one of the following two statements holds:

a) There exists $x \in \mathbb{R}^n$ such that $Ax = b$ and $x \geq 0$

b) There exists $\mathrm{y} \in \mathbb{R}^m$ such that $A^T y \geq 0$ and $b^T y < 0$

Claim: if (a) does not hold, then (b) must hold.

- Consider $\mathrm{Z} = \{Ax: x \geq 0\}$ so that Z is closed and convex
- (a) does not hold $\Leftrightarrow$ $b \notin Z$
- By separating hyperplane theorem, there exists hyperplane $\alpha \cdot z = \beta$ such that $\alpha^T \cdot z \geq \beta$ for all $z \in Z$ and $\alpha^T \cdot b < \beta$
- Note $0 \in Z$, therefore $\beta \leq \alpha^T \cdot 0 = 0$ and thus $\alpha^T \cdot b < 0$
- $\alpha^T A x \geq \beta$ for any $x \geq 0$ implies $\alpha^T A \geq 0$ since x can be arbitrary large
- Letting α be our y yields the lemma

An Alternative of Farkas' Lemma

Following corollary of Farkas' lemma is more convenient for our proof

Corollary: Exactly one of the following systems holds:

$\exists\, x \in \mathbb{R}^n$, s.t.

$$A \cdot x \leq b$$
$$x \geq 0$$

$\exists\, y \in \mathbb{R}^m$, s.t.

$$A^t \cdot y \geq 0$$
$$b^t \cdot y < 0$$
$$y \geq 0$$

Compare to the original version

$\exists\, x \in \mathbb{R}^n$, s.t.

$$A \cdot x = b$$
$$x \geq 0$$

$\exists\, y \in \mathbb{R}^m$, s.t.

$$A^t \cdot y \geq 0$$
$$b^t \cdot y < 0$$

An Alternative of Farkas' Lemma

Following corollary of Farkas' lemma is more convenient for our proof

Corollary: Exactly one of the following systems holds:

$\exists\, x \in \mathbb{R}^n$, s.t.

$$A \cdot x \leq b$$
$$x \geq 0$$

$\exists\, y \in \mathbb{R}^m$, s.t.

$$A^t \cdot y \geq 0$$
$$b^t \cdot y < 0$$
$$y \geq 0$$

Proof: Apply Fakas' lemma to the following linear systems

$\exists\, x \in \mathbb{R}^n$, s.t.

$$A \cdot x + I \cdot s = b$$
$$x, s \geq 0$$

$\exists\, y \in \mathbb{R}^m$, s.t.

$$A^t \cdot y \geq 0$$
$$I \cdot y \geq 0$$
$$b^t \cdot y < 0$$

Proof of Strong Duality

Primal LP

max	$c^t \cdot x$
s.t.	$Ax \leq b$
	$x \geq 0$

Dual LP

min	$b^t \cdot y$
s.t.	$A^t y \geq c$
	$y \geq 0$

Theorem [Strong Duality]: If either the primal or dual is feasible and bounded, then so is the other and OPT(primal) = OPT(dual).

Proof

- Dual of the dual is primal; so w.l.o.g assume primal is feasible and bounded
- Weak duality yields OPT(primal) ≤ OPT(dual)
- Next we prove the converse, i.e., OPT(primal) ≥ OPT(dual)

Proof of Strong Duality

Primal LP

max $c^t \cdot x$

s.t. $Ax \leq b$

$x \geq 0$

Dual LP

min $b^t \cdot y$

s.t. $A^t y \geq c$

$y \geq 0$

- We prove if OPT(primal)< β for some β, then OPT(dual)< β
- Apply Farkas' lemma to the following linear system

$\exists x \in \mathbb{R}^n$ such that

$Ax \leq b$

$-c^t \cdot x \leq -\beta$

$x \geq 0$

$\exists y \in \mathbb{R}^m$ and $z \in \mathbb{R}$

$A^t y - cz \geq 0$

$b^T y - \beta z < 0$

$y, z \geq 0$

- By assumption, the first system is infeasible, so the second must hold
  - If $z > 0$, can rescale (y, z) to make $z = 1$, yielding OPT(dual)< β
  - If $z = 0$, then system $A^t y \geq 0, b^T y < 0, y \geq 0$ feasible. Fakas' lemma implies that system $Ax \leq b, x \geq 0$ is infeasible, contradicting theorem assumption.

Outline

- Recap and Weak Duality
- Strong Duality and Its Proof
- Consequence of Strong Duality

Complementary Slackness

Primal LP

max	$c^t \cdot x$
s.t.	$Ax \leq b$
	$x \geq 0$

Dual LP

min	$b^t \cdot y$
s.t.	$A^t y \geq c$
	$y \geq 0$

- $s_i = (b - Ax)_i$ is the i'th primal slack variable
- $t_j = (A^T y - c)_j$ is the j'th dual slack variable

Complementary Slackness:

x and y are optimal if and only if they are feasible and

- $x_j t_j = 0$ for all j $= 1, \cdots, m$
- $y_i s_i = 0$ for all $i = 1, \cdots, n$

Remark: can be used to recover optimal solution of the primal from optimal solution of the dual (very useful in optimization).

Economic Interpretation of Complementary Slackness:

Given the optimal production and optimal raw material prices

- It only produces products for which profit equals raw material cost
- A raw material is priced greater than 0 only if it is used up in the optimal production

Primal LP

max $c^T \cdot x$

s.t. $\sum_{j=1}^{n} a_{ij}\, x_j \le b_i, \quad \forall i \in [m]$

$x_j \ge 0, \quad \forall j \in [n]$

Dual LP

min $b^T \cdot y$

s.t. $\sum_{i=1}^{m} a_{ij}\, y_i \ge c_j, \quad \forall j \in [n]$

$y_i \ge 0, \quad \forall i \in [m]$

Proof of Complementary Slackness

Primal LP

max $c^t \cdot x$

s.t. $Ax \leq b$

$x \geq 0$

Dual LP

min $b^t \cdot y$

s.t. $A^t y \geq c$

$y \geq 0$

Proof of Complementary Slackness

Primal LP

max	$c^t \cdot x$
s.t.	$Ax + s = b$
	$x, s \geq 0$

Dual LP

min	$b^t \cdot y$
s.t.	$A^t y - t = c$
	$y, t \geq 0$

- Add slack variables into both LPs

$$y^T b - x^T c = y^T(Ax + s) - x^T(A^T y - t) = y^T s + x^T t$$

- For any feasible x, y, the gap between primal and dual objective value is precisely the "aggregated slackness" $y^T s + x^T t$
- Strong duality implies $y^T s + x^T t = 0$ for the optimal x, y.
- Since $x, s, y, t \geq 0$, we have $x_j t_j = 0$ for all j and $y_i s_i = 0$ for all i.

Thank You

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