

CS650 I: Topics in Learning and Game Theory (Spring 2021)

Introduction to Game Theory (I)

Instructor: Haifeng Xu

Outline

- Games and its Basic Representation
- Nash Equilibrium and its Computation
- Other (More General) Classes of Games

(Recall) Example 1: Prisoner's Dilemma

- Two members A,B of a criminal gang are arrested
- They are questioned in two separate rooms
 - ❖ No communications between them

		B	
		B stays silent	B betrays
A	A stays silent	-1, -1	-3, 0
	A betrays	0, -3	-2, -2

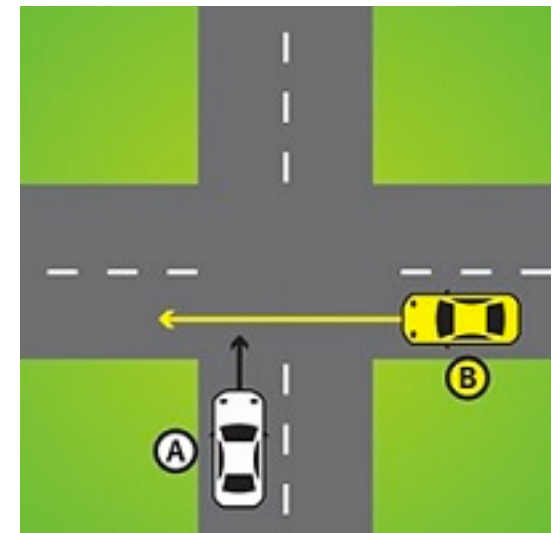
Q: How should each prisoner act?

- Both of them betray, though (-1,-1) is better for both

Example 2: Traffic Light Game

- Two cars heading to orthogonal directions

		B	
		STOP	GO
A	STOP	$(-3, -2)$	$(-3, 0)$
	GO	$(0, -2)$	$(-100, -100)$



Q: what are the equilibrium statuses?

Answer: (STOP, GO) and (GO, STOP)

Example 3: Rock-Paper-Scissor

		Player 2		
		Rock	Paper	Scissor
Player 1	Rock	(0, 0)	(-1, 1)	(1, -1)
	Paper	(1, -1)	(0, 0)	(-1, 1)
	Scissor	(-1, 1)	(1, -1)	(0, 0)

Q: what is an equilibrium?

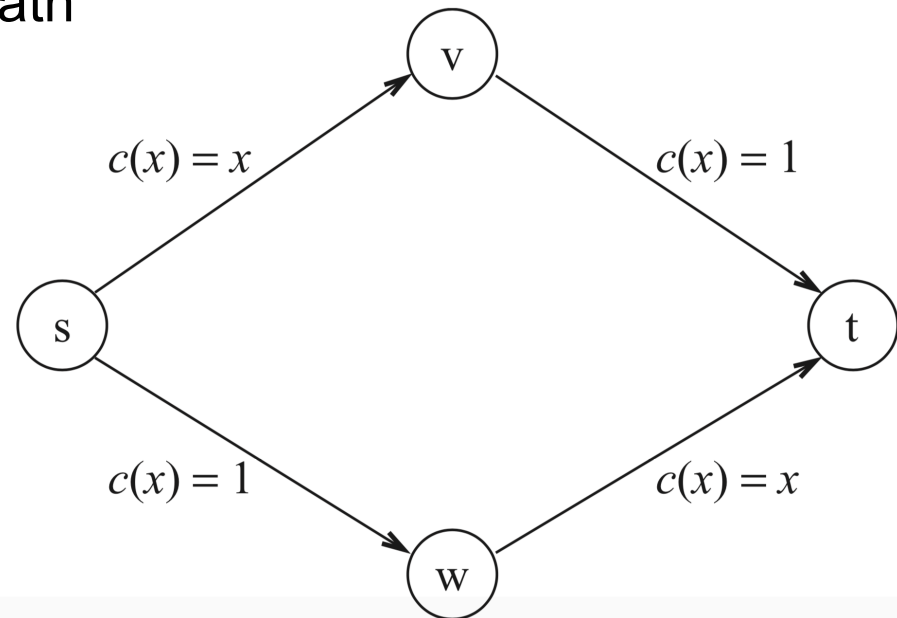
- Need to randomize – any deterministic action pair cannot make both players happy
- Common sense suggests $(1/3, 1/3, 1/3)$

Example 4: Selfish Routing

- One unit flow from s to t which consists of (infinite) individuals, each controlling an infinitesimal small amount of flow
- Each individual wants to minimize his own travel time

Q: What is the equilibrium status?

- Half unit flow through each path
- **Social cost** = $3/2$

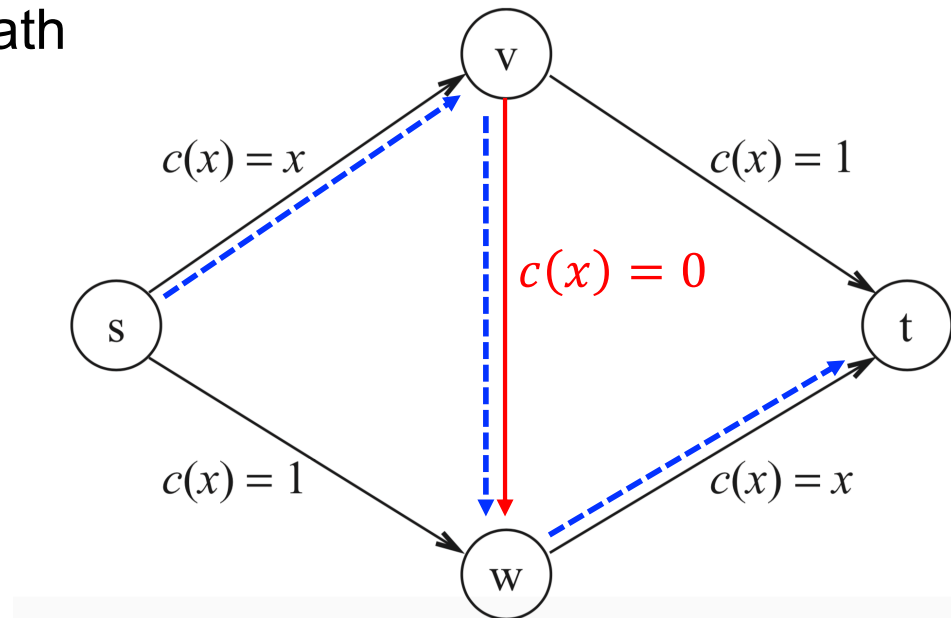


Example 4: Selfish Routing

- One unit flow from s to t which consists of (infinite) individuals, each controlling an infinitesimal small amount of flow
- Each individual wants to minimize his own travel time

Q: What is the equilibrium status after adding a superior high way with 0 traveling cost?

- Everyone takes the blue path
- Social cost = 2



Key Characteristics of These Games

- Each agent wants to maximize her own payoff
- An agent's payoff depends on other agents' actions
- The interaction stabilizes at a state where no agent can increase his payoff via **unilateral deviation**

Strategic Games Are Ubiquitous

➤ Pricing

☐ Spirit Airlines (2) \$438

Departure time - Boston

☐ Morning (5:00am - 11:59am)

☐ Afternoon (12:00pm - 5:59pm)

☐ Evening (6:00pm - 11:59pm)

Arrival time - Chicago

☐ Early Morning (12:00am - 4:59am)

☐ Morning (5:00am - 11:59am)

☐ Afternoon (12:00pm - 5:59pm)

☐ Evening (6:00pm - 11:59pm)

6:30am - 8:15am

United

Very Good Flight (8.1/10)

[Details & baggage fees](#)

9:23am - 11:27am

American Airlines

Very Good Flight (8.3/10)

[Details & baggage fees](#)

7:01am - 9:10am

American Airlines

Very Good Flight (8.3/10)

[Details & baggage fees](#)

5:30am - 8:50am

Delta

Satisfactory Flight (6.4/10)

[Details & baggage fees](#)

2h 45m (Nonstop)

BOS - ORD

5 left at **\$236**

roundtrip

Select

3h 4m (Nonstop)

BOS - ORD

\$236

roundtrip

Select

3h 9m (Nonstop)

BOS - ORD

\$236

roundtrip

Select

4h 20m (1 stop)

BOS - 42m in DTW - ORD

1 left at **\$246**

roundtrip

Select

Run this search to

Strategic Games Are Ubiquitous

- Pricing
- Sponsored search
 - Drives 90%+ of Google's revenue

Google search results for "where to buy cruise vacation". The search bar shows the query and the Google logo. Below the search bar, there are tabs for "All", "Shopping", "Images", "News", "Videos", "More", "Settings", and "Tools". The results show "About 103,000,000 results (0.63 seconds)".

The first sponsored result is from Carnival Cruise Line, highlighted with a red box. It includes the URL www.carnival.com/ and a price of **\$1.03**. The text below the URL reads: "Make Your Vacation Dreams A Reality With A Carnival® Cruise. Book Online Today! Signature Dining. 2-5 Day Cruises Set Sail On These Quick Getaways That Fit Any Calendar, Anytime. 6-9 Day Cruises Full-Length Cruises Mean More Time For Sun-Soaked Relaxation And Fun."

The second sponsored result is from Expedia Cruises | Cruise Vacations, with the URL www.expedia.com/Cruises and a price of **\$1.02**. The text below the URL reads: "Find the Perfect Cruise at the Best Price on Expedia, the #1 Travel Website. Last Minute Cruise Deals. Best Price Guaranteed. 4,000 Cruises Worldwide. Luxury Cruises Available. Destinations: Caribbean, Bahamas, Alaska, Mexico, Europe, Bermuda, Hawaii, Canada/New England."

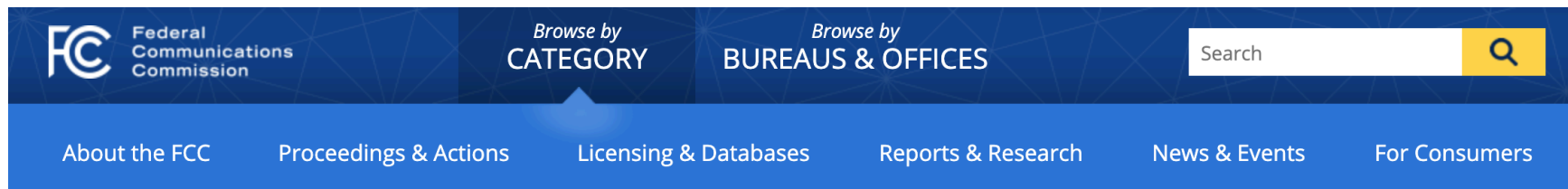
The third sponsored result is from 2019 Cruises 82% Off | Compare All Cruise Lines | VacationsToGo.com, with the URL www.vacationstogo.com/ and a price of **\$0.60**. The text below the URL reads: "Book today for best price and selection on 2019 cruises. Save up to 82% Off on Every Ship. Last-Minute Cruise Deals · Age 55+ Discounts · Caribbean up to 82% Off · Huge Carnival Deals"

The fourth sponsored result is from KAYAK® Cruise Search | Find the Cheapest Cruise Deals | kayak.com, with the URL www.kayak.com/vacations-go/last and a price of **\$0.21**. The text below the URL reads: "Rating for kayak.com: 3.8 - 605 reviews"

On the right side of the search results, there is a sponsored section titled "See cruise vac..." with a "Sponsored" label. It features an image of a cruise ship and a price of **\$0.65** for a "3-D Cruise Ship Centerpiece Zoom Party". The original price of \$6.65 is crossed out. Below this section is a link "→ More on Google".

Strategic Games Are Ubiquitous

- Pricing
- Sponsored search
 - Drives 90%+ of Google's revenue
- FCC's Allocation of spectrum to radio frequency users



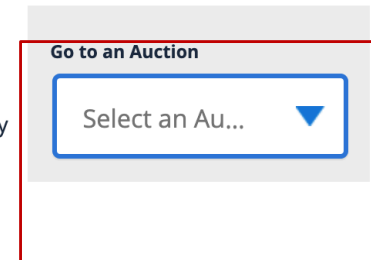
Home / Economics and Analytics /

Auctions

Proceedings & Actions

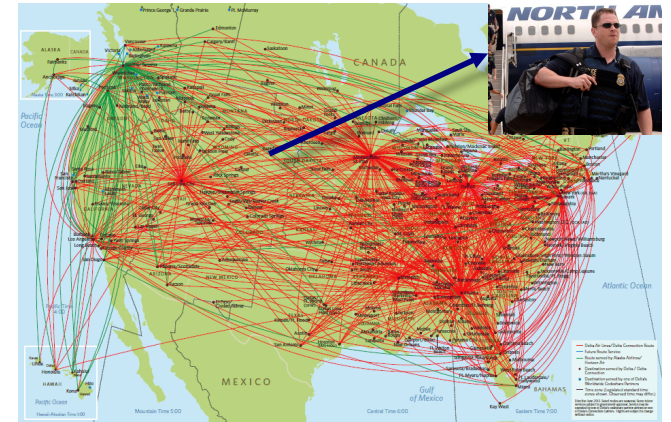
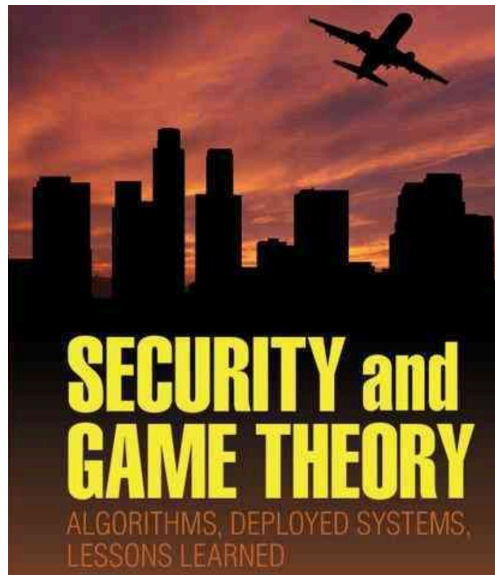
Proceedings and Actions
Overview

Since 1994, the Federal Communications Commission (FCC) has conducted auctions of licenses for electromagnetic spectrum. These auctions are open to any eligible company or individual that submits an application and upfront payment, and is found to be a qualified bidder by the Commission ([More About Auctions...](#))



Strategic Games Are Ubiquitous

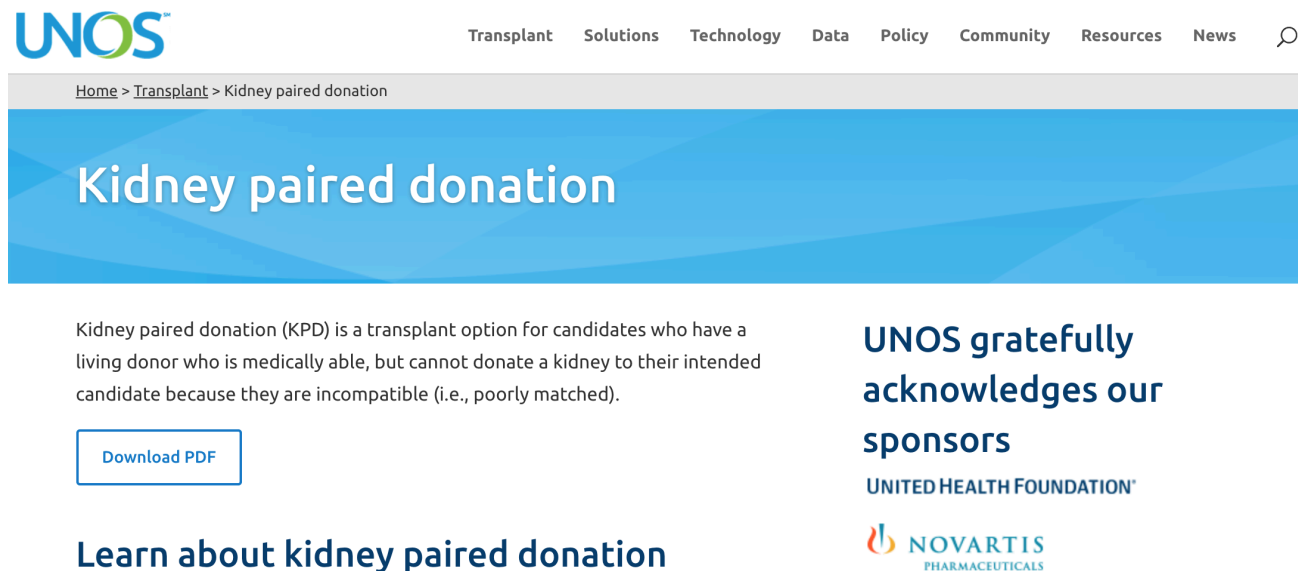
- Pricing
- Sponsored search
 - Drives 90%+ of Google's revenue
- FCC's Allocation of spectrum to radio frequency users
- National security, boarder patrolling, counter-terrorism



Optimize resource allocation against
attackers/adversaries

Strategic Games Are Ubiquitous

- Pricing
- Sponsored search
 - Drives 90%+ of Google's revenue
- FCC's Allocation of spectrum to radio frequency users
- National security, boarder patrolling, counter-terrorism
- Kidney exchange – decides who gets which kidney at when



The screenshot shows the UNOS website's page for "Kidney paired donation". The header includes the UNOS logo and a navigation menu with links: Transplant, Solutions, Technology, Data, Policy, Community, Resources, and News. Below the header is a breadcrumb trail: Home > Transplant > Kidney paired donation. The main heading "Kidney paired donation" is displayed in large white text on a blue background. Below this, a paragraph explains that Kidney paired donation (KPD) is a transplant option for candidates who have a living donor who is medically able, but cannot donate a kidney to their intended candidate because they are incompatible (i.e., poorly matched). A "Download PDF" button is provided. To the right, a section titled "UNOS gratefully acknowledges our sponsors" lists the "UNITED HEALTH FOUNDATION" and "NOVARTIS PHARMACEUTICALS". At the bottom left, a link says "Learn about kidney paired donation".

UNOS

Transplant Solutions Technology Data Policy Community Resources News

Home > Transplant > Kidney paired donation

Kidney paired donation

Kidney paired donation (KPD) is a transplant option for candidates who have a living donor who is medically able, but cannot donate a kidney to their intended candidate because they are incompatible (i.e., poorly matched).

[Download PDF](#)

[Learn about kidney paired donation](#)

UNOS gratefully acknowledges our sponsors

UNITED HEALTH FOUNDATION*

NOVARTIS
PHARMACEUTICALS

Strategic Games Are Ubiquitous

- Pricing
- Sponsored search
 - Drives 90%+ of Google's revenue
- FCC's Allocation of spectrum
- National security, border patrol
- Kidney exchange – decide who gets which
- Entertainment games: poker, blackjack, Go, chess . . .
- Social choice problems such as voting, fair division, etc.



Strategic Games Are Ubiquitous

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- Entertainment games: poker, blackjack, Go, chess . . .
- Social choice problems such as voting, fair division, etc.

These are just a few example domains *where computer science has made significant impacts*; There are many others.

Main Components of a Game

- **Players**: participants of the game, each may be an individual, organization, a machine or an algorithm, etc.
- **Strategies**: actions available to each player
- **Outcome**: the profile of player strategies
- **Payoffs**: a function mapping an outcome to a utility for each player

Normal-Form Representation

- n players, denoted by set $[n] = \{1, \dots, n\}$
- Player i takes action $a_i \in A_i$
- An outcome is the **action profile** $a = (a_1, \dots, a_n)$
 - As a convention, $a_{-i} = (a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n)$ denotes all actions excluding a_i
- Player i receives payoff $u_i(a)$ for any outcome $a \in \prod_{i=1}^n A_i$
 - $u_i(a) = u_i(a_i, a_{-i})$ depends on other players' actions
- $\{A_i, u_i\}_{i \in [n]}$ are public knowledge

This is the most basic game model

- There are game models with richer and more intricate structures

Illustration: Prisoner's Dilemma

- 2 players: 1 and 2
- $A_i = \{\text{silent}, \text{betray}\}$ for $i = 1, 2$
- An outcome can be, e.g., $a = (\text{silent}, \text{silent})$
- $u_1(a), u_2(a)$ are pre-defined, e.g., $u_1(\text{silent}, \text{silent}) = -1$
- The whole game is public knowledge; players take actions simultaneously
 - Equivalently, take actions without knowing the others' actions

Dominant Strategy

An action a_i is a **dominant strategy** for player i if a_i is better than any other action $a'_i \in A_i$, **regardless** what actions other players take.

Formally,

$$u_i(a_i, a_{-i}) \geq u_i(a'_i, a_{-i}), \quad \forall a'_i \neq a_i \text{ and } \forall a_{-i}$$

Note: “strategy” is just another term for “action”

A \ B	B stays silent	B betrays
A stays silent	-1, -1	-3, 0
A betrays	0, -3	-2, -2

Prisoner's Dilemma

- *Betray* is a dominant strategy for both
- Dominant strategies do not always exist
 - For example, the traffic light game

	STOP	GO
STOP	(-3, -2)	(-3, 0)
GO	(0, -2)	(-100, -100)

Equilibrium

- An outcome a^* is an equilibrium if no player has incentive to deviate **unilaterally**. More formally,

$$u_i(a_i^*, a_{-i}^*) \geq u_i(a_i, a_{-i}^*), \quad \forall a_i \in A_i$$

- A special case of Nash Equilibrium, a.k.a., **pure strategy NE**
- If each player has a dominant strategy, they form an equilibrium
- But, an equilibrium does not need to consist of dominant strategies

		B	
		STOP	GO
A	STOP	(-3, -2)	(-3, 0)
	GO	(0, -2)	(-100, -100)

Traffic Light Game

Equilibrium

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- If each player has a dominant strategy, they form an equilibrium
- But, an equilibrium does not need to consist of dominant strategies

Pure strategy NE does not always exist...

	Rock	Paper	Scissor
Rock	(0, 0)	(-1, 1)	(1, -1)
Paper	(1, -1)	(0, 0)	(-1, 1)
Scissor	(-1, 1)	(1, -1)	(0, 0)

Outline

- Games and its Basic Representation
- Nash Equilibrium and its Computation
- Other (More General) Classes of Games

Pure vs Mixed Strategy

- Pure strategy: take an action deterministically
- Mixed strategy: can randomize over actions
 - Described by a distribution x_i where $x_i(a_i) = \text{prob. of taking action } a_i$
 - $|A_i|$ -dimensional simplex $\Delta_{A_i} := \{x_i: \sum_{a_i \in A_i} x_i(a_i) = 1, x_i(a_i) \geq 0\}$ contains all possible mixed strategies for player i
 - Players draw their own actions *independently*
- Given strategy profile $x = (x_1, \dots, x_n)$, expected utility of i is

$$\sum_{a \in A} u_i(a) \cdot \prod_{i \in [n]} x_i(a_i)$$

- Often denoted as $u(x)$ or $u(x_i, x_{-i})$ or $u(x_1, \dots, x_n)$
- When x_i corresponds to some pure strategy a_i , we also write $u(a_i, x_{-i})$
- Fix x_{-i} , $u(x_i, x_{-i})$ is **linear** in x_i

Best Responses

Fix any x_{-i} , x_i^* is called a best response to x_{-i} if

$$u_i(x_i^*, x_{-i}) \geq u_i(x_i, x_{-i}), \quad \forall x_i \in \Delta_{A_i}.$$

Claim. There always exists a pure best response

Proof: linear program “max $u_i(x_i, x_{-i})$ subject to $x_i \in \Delta_{A_i}$ ” has a vertex optimal solution

Remark: If x_i^* is a best response to x_{-i} , then any a_i in the support of x_i^* (i.e., $x_i^*(a_i) > 0$) must be equally good and are all “pure” best responses

Nash Equilibrium (NE)

A mixed strategy profile $x^* = (x_1^*, \dots, x_n^*)$ is a **Nash equilibrium** if

$$u_i(x_i^*, x_{-i}^*) \geq u_i(x_i, x_{-i}^*), \quad \forall x_i \in \Delta_{A_i}, \forall i \in [n].$$

That is, for any i , x_i^* is a best response to x_{-i}^* .

Remarks

- An equivalent condition: $u_i(x_i^*, x_{-i}^*) \geq u_i(a_i, x_{-i}^*), \forall a_i \in A_i, \forall i \in [n]$
 - Since there always exists a pure best response
- It is not clear yet that such a mixed strategy profile would exist
 - Recall that pure strategy Nash equilibrium may not exist

Nash Equilibrium (NE)

Theorem (Nash, 1951): Every finite game (i.e., finite players and actions) admits at least one mixed strategy Nash equilibrium.

- A foundational result in game-theory
- Example: rock-paper-scissor – what is a mixed strategy NE?
 - $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is a best response to $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$

		1/3	1/3	1/3
		Rock	Paper	Scissor
ExpU = 0	Rock	(0, 0)	(-1, 1)	(1, -1)
ExpU = 0	Paper	(1, -1)	(0, 0)	(-1, 1)
ExpU = 0	Scissor	(-1, 1)	(1, -1)	(0, 0)

Nash Equilibrium (NE)

Theorem (Nash, 1951): Every finite game (i.e., finite players and actions) admits at least one mixed strategy Nash equilibrium.

- An equilibrium outcome is not necessarily the best for players
 - Equilibrium only describes where the game stabilizes at
 - Many researches on understanding how self-interested behaviors reduces overall social welfare (recall the selfish routing game)
- A game may have many, even infinitely many, NEs
 - The issue of **equilibrium selection**

A \ B	B	
	B stays silent	B betrays
A stays silent	-1 / -1	0 / -3
A betrays	0 / -3	-2 / -2

Intractability of Finding a NE

Theorem: Computing a Nash equilibrium for any two-player normal-form game is PPAD-hard.

Note: PPAD-hard problems are believed to not admit poly time algorithm

- A two player game can be described by $2mn$ numbers – $u_1(i, j)$ and $u_2(i, j)$ where $i \in [m]$ is player 1's action and $j \in [n]$ is player 2's.
- Theorem implies no $\text{poly}(mn)$ time algorithm to compute an NE for any input game
- Ok, so what can we hope?
 - If the game has good structures, maybe we can find an NE efficiently
 - For example, zero-sum ($u_1(i, j) + u_2(i, j) = 0$ for all i, j), some resource allocation games

An Exponential-Time Alg for Two-Player Nash

- What if we know the support of the NE: S_1, S_2 for player 1 and 2?
- The NE can be formulated by a **linear feasibility** problem with variables x_1^*, x_2^*, U_1, U_2

$$\begin{aligned}\forall j \in S_2: & \quad \sum_{i \in S_1} u_2(i, j) x_1^*(i) = U_2 \\ \forall j \notin S_2: & \quad \sum_{i \in S_1} u_2(i, j) x_1^*(i) \leq U_2\end{aligned}$$

- The challenge of computing a NE is to find the correct supports
 - No general tricks, typically just try all possibilities
 - Some pre-processing may help, e.g., eliminating dominated actions
- This approach does not work for > 2 players games (why?)

Intractability of Finding “Best” NE

Theorem: It is NP-hard to compute the NE that maximizes the sum of players' utilities or any single player's utility even in two-player games.

- Proofs of these results for NEs are beyond the scope of this course

Outline

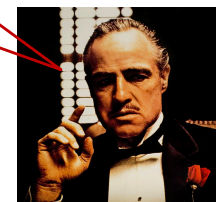
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Bayesian Games

- Previously, assumed players have complete knowledge of the game
- What if players are uncertain about the game?
- Can be modeled as a Bayesian belief about the state of the game
 - This is typical in Bayesian decision making, but not the only way

		B	
		B stays silent	B betrays
A	A stays silent	$\theta - 1$ $-1 + \theta$	$-3 + \theta$ $-3 + \theta$
	A betrays	$\theta - 3$ 0	-2 -2

I will give an additional reward θ for whoever staying silent



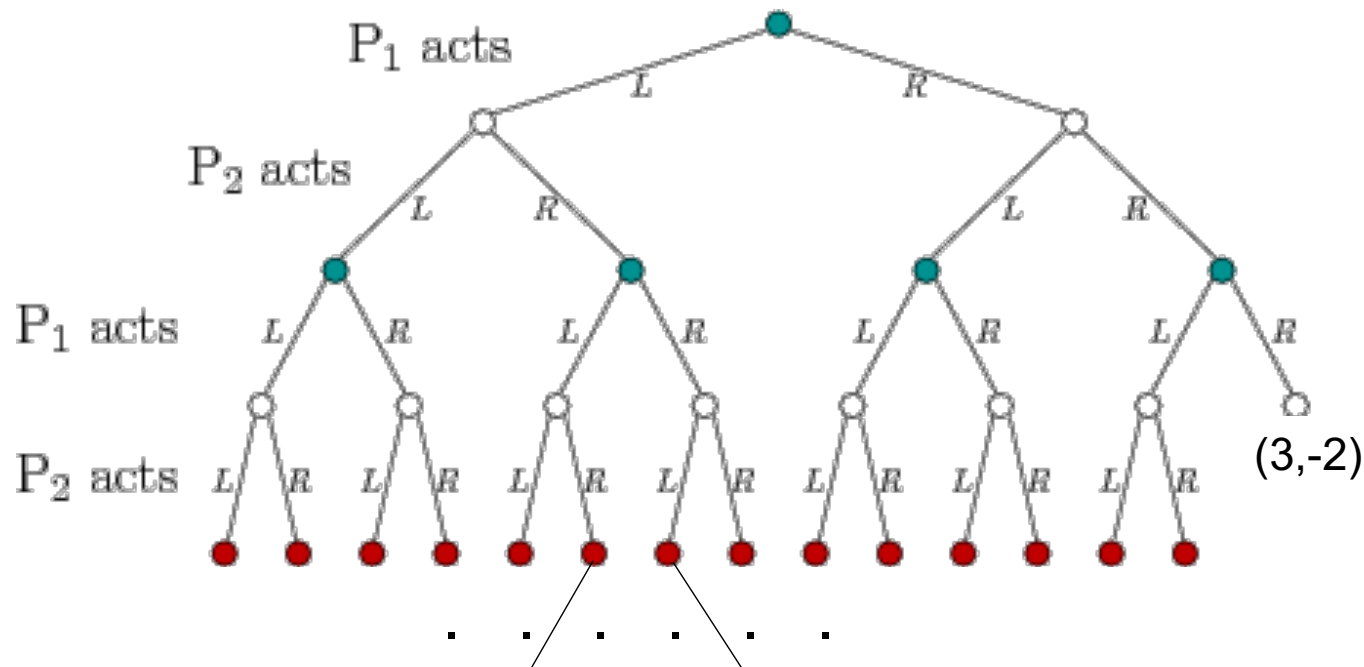
- It is believed that $\theta \in \{0, 2, 4\}$ uniformly at random
- Or maybe the two players have different beliefs about θ

Bayesian Games

- Previously, assumed players have complete knowledge of the game
- What if players are uncertain about the game?
- Can be modeled as a Bayesian belief about the state of the game
 - This is typical in Bayesian decision making, but not the only way
- More generally, can model player i ' payoffs as u_i^θ where θ is a **random** state of the game
- Each player obtains a (random) signal s_i that is correlated with θ
 - A joint prior distribution over $(\theta, s_1, \dots, s_n)$ is assumed the public knowledge
- Can define a similar notion as Nash equilibrium, but expected utility also incorporates the randomness of the state of the game θ
- Applications: poker, blackjack, auction design, etc.

Extensive-Form Games (EFGs)

- Previously, assumed players move only once and **simultaneously**
- More generally, can move sequentially and for multiple rounds
- Modeled by extensive-form game, described by a **game tree**



Extensive-Form Games (EFGs)

- Previously, assumed players move only once and **simultaneously**
- More generally, can move sequentially and for multiple rounds
- Modeled by extensive-form game, described by a **game tree**
- EFGs are extremely general, can represent almost all kinds of games, but of course very difficult to solve

A Remark

Sequential move fundamentally differs from simultaneous move

Nash equilibrium is only for simultaneous move

A Remark

Sequential move fundamentally differs from simultaneous move

Nash equilibrium is only for simultaneous move

- What is an NE?
 - (a_2, b_2) is the unique Nash, resulting in utility pair (1,2)
- If A moves first; B sees A's move and then best responds, how should A play?
 - Play action a_1 deterministically!

		B	
		b_1	b_2
A	a_1	(2, 1)	(-2, -2)
	a_2	(2.01, -2)	(1, 2)

This sequential game model is called **Stackelberg game**, originally used to model market competition and now adversarial attacks.

Thank You

Haifeng Xu

University of Virginia

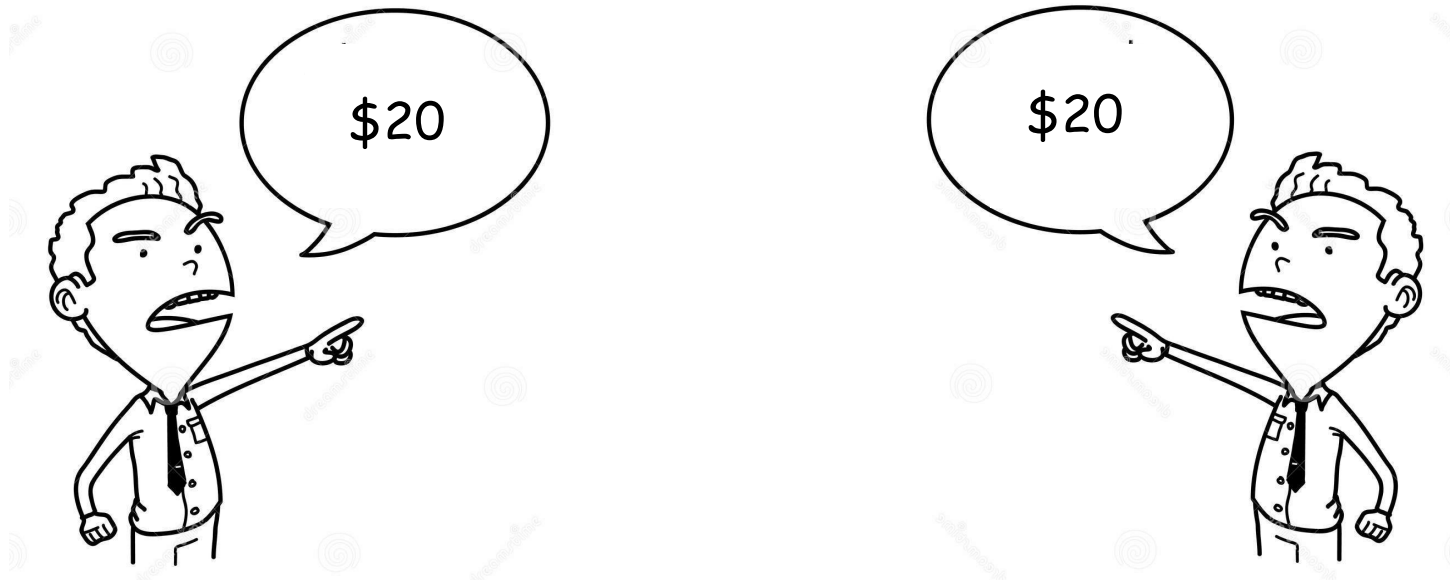
hx4ad@virginia.edu

Appendix

Recall: Competing Book Sellers

- Assume people will buy if the book price $\leq \$200$
- Product cost = \$20
- Two book sellers compete for customers

Q: what price should each seller set?



Recall: Competing Book Sellers

- Assume people will buy if the book price $\leq \$200$
- Product cost = \$20
- Two book sellers compete for customers

Q: what price should each seller set?

- The market reaches a “stable status” (a.k.a., equilibrium)
- Nobody can benefit via *unilateral deviation*

