Announcements

- HW 1 is out, due March 5'th, 6 pm

CS6501: Topics in Learning and Game Theory
(Spring 2021)

Introduction to Game Theory (II)

Instructor: Haifeng Xu

Outline

- Correlated and Coarse Correlated Equilibrium
- Zero-Sum Games

Recap: Normal-Form Games

- n players, denoted by set $[n] = \{1, \cdots, n\}$
- Player i takes action $a_i \in A_i$
- An outcome is the action profile $a = (a_1, \cdots, a_n)$
  - As a convention, $a_{-i} = (a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_n)$ denotes all actions excluding a_i
- Player i receives payoff $u_i(a)$ for any outcome $a \in \Pi_{i=1}^n A_i$
  - $u_i(a) = u_i(a_i, a_{-i})$ depends on other players' actions
- $\{A_i, u_i\}_{i \in [n]}$ are public knowledge

A mixed strategy profile $x^* = (x_1^*, \cdots, x_n^*)$ is a **Nash equilibrium (NE)** if for any i, x_i^* is a best response to x_{-i}^*.

NE Is Not the Only Solution Concept

- NE rests on two key assumptions
  1. Players move simultaneously (so they cannot see others' strategies before the move)

Sequential move fundamentally differs from simultaneous move

An Example

- What is an NE?
  - (a_2, b_2) is the unique Nash, resulting in utility pair (1,2)
- If A moves first; B sees A's move and then best responds, how should A play?
  - Play action a_1 deterministically!

B

		b_1	b_2
A	a_1	(2, 1)	(-2, -2)
	a_2	(2.01, -2)	(1, 2)

This sequential game model is called Stackelberg game, its equilibrium is called Strong Stackelberg equilibrium

An Example

When is sequential move more realistic?

- Market competition: market leader (e.g., Facebook) vs competing followers (e.g., small start-ups)
- Adversarial attacks: a learning algorithm vs an adversary, security agency vs real attackers
  - ✓ Used a lot in recent adversarial ML literature

This is precisely the reason that we need different equilibrium concepts to model different scenarios.

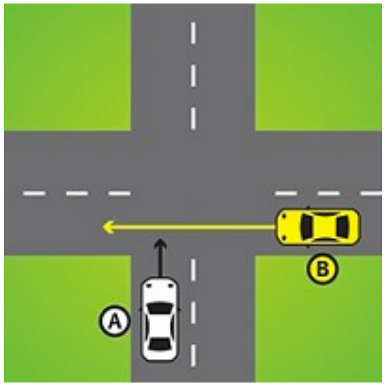
NE Is Not the Only Solution Concept

- NE rests on two key assumptions
  1. Players move simultaneously (so they cannot see others' strategies before the move)
  2. Players take actions independently

Today: we study what happens if players do not take actions independently but instead are "coordinated" by a central mediator

- This results in the study of correlated equilibrium

An Illustrative Example



B

		STOP	GO
A	STOP	(-3, -2)	(-3, 0)
	GO	(0, -2)	(-100, -100)

The Traffic Light Game

Well, we did not see many crushes in reality… Why?

- There is a mediator – the traffic light – that coordinates cars' moves
- For example, recommend (GO, STOP) for (A,B) with probability 3/5 and (STOP, GO) for (A,B) with probability 2/5
  - GO = green light, STOP = red light
  - Following the recommendation is a best response for each player
  - It turns out that this recommendation policy results in equal player utility – $6/5$ and thus is "fair"

This is exactly how traffic lights are designed!

Correlated Equilibrium (CE)

- A (randomized) recommendation policy π assigns probability $\pi(a)$ for each action profile $a \in A = \Pi_{i \in [n]} A_i$
  - A mediator first samples $a \sim \pi$, then recommends a_i to i *privately*
- Upon receiving a recommendation a_i, player i's expected utility is

$$\frac{1}{c} \sum_{a_{-i} \in A_{-i}} u_i(a_i, a_{-i}) \cdot \pi(a_i, a_{-i})$$

  - c is a normalization term that equals the probability a_i is recommended

A recommendation policy π is a **correlated equilibrium** if

$$\sum_{a_{-i}} u_i(a_i, a_{-i}) \cdot \pi(a_i, a_{-i}) \geq \sum_{a_{-i}} u_i(a'_i, a_{-i}) \cdot \pi(a_i, a_{-i}), \forall\, a'_i \in A_i, \forall i \in [n].$$

- That is, any recommended action to any player is a best response
  - CE makes incentive compatible action recommendations
- Assumed π is public knowledge so every player can calculate her utility

Basic Facts about Correlated Equilibrium

Fact. Any Nash equilibrium is also a correlated equilibrium.

- True by definition. Nash equilibrium can be viewed as independent action recommendation
- As a corollary, correlated equilibrium always exists

Fact. The set of correlated equilibria forms a convex set.

- In fact, distributions π satisfies a set of linear constraints

$$\sum_{a_{-i}} u_i(a_i, a_{-i}) \cdot \pi(a_i, a_{-i}) \geq \sum_{a_{-i}} u_i({a'}_i, a_{-i}) \cdot \pi(a_i, a_{-i}), \forall\, {a'}_i \in A_i, \forall i \in [n].$$

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Fact. The set of correlated equilibria forms a convex set.

- In fact, distributions π satisfies a set of linear constraints
- This is nice because that allows us to optimize over all CEs
- Not true for Nash equilibrium

Coarse Correlated Equilibrium (CCE)

- A weaker notion of correlated equilibrium
- Also a recommendation policy π, but only requires that any player does not have incentives to opting out of our recommendations

A recommendation policy π is a **coarse correlated equilibrium** if

$$\sum_{a \in A} u_i(a) \cdot \pi(a) \geq \sum_{a \in A} u_i(a'_i, a_{-i}) \cdot \pi(a), \forall a'_i \in A_i, \forall i \in [n].$$

That is, for any player i, following π's recommendations is better than opting out of the recommendation and “acting on his own”.

Compare to correlated equilibrium condition:

$$\sum_{a_{-i}} u_i(a_i, a_{-i}) \cdot \pi(a_i, a_{-i}) \geq \sum_{a_{-i}} u_i(a'_i, a_{-i}) \cdot \pi(a_i, a_{-i}), \forall a'_i \in A_i, \forall i \in [n].$$

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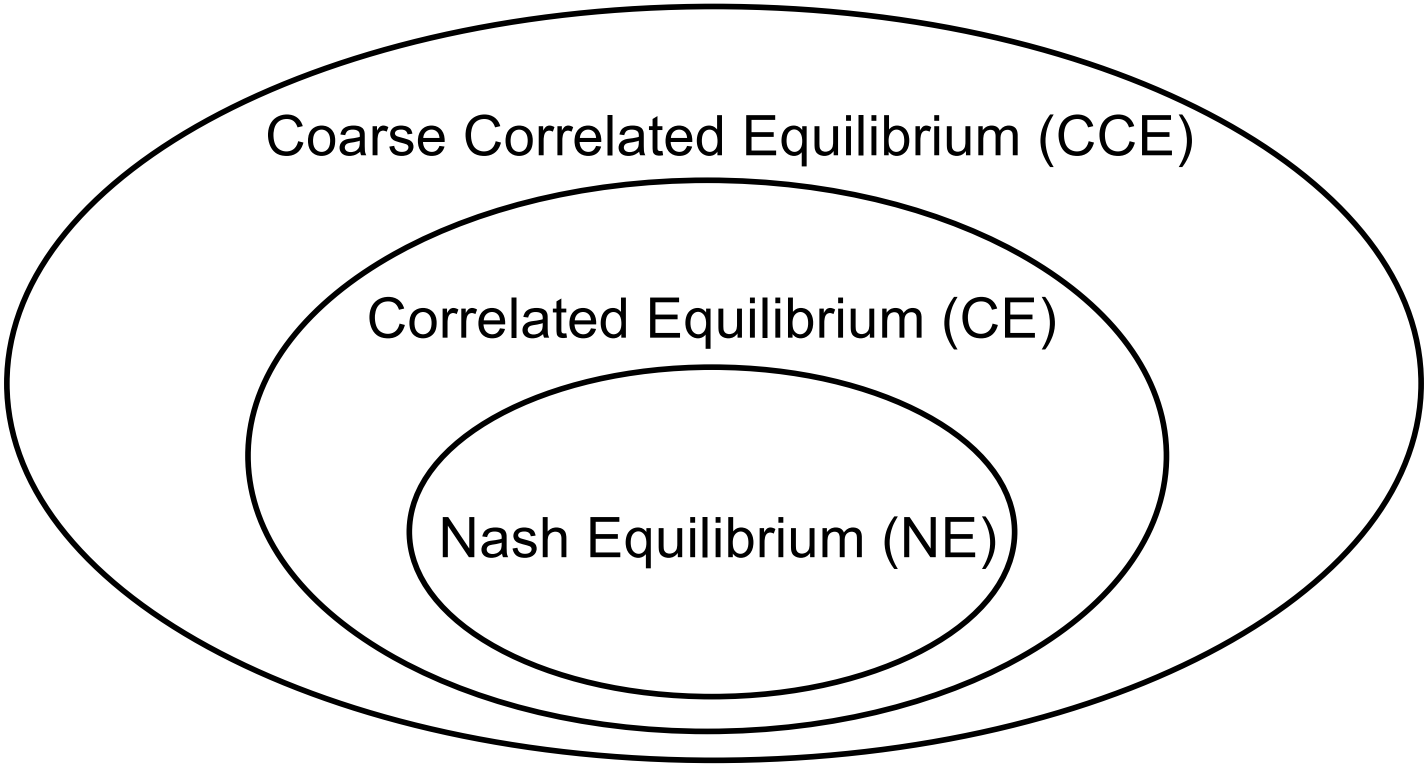
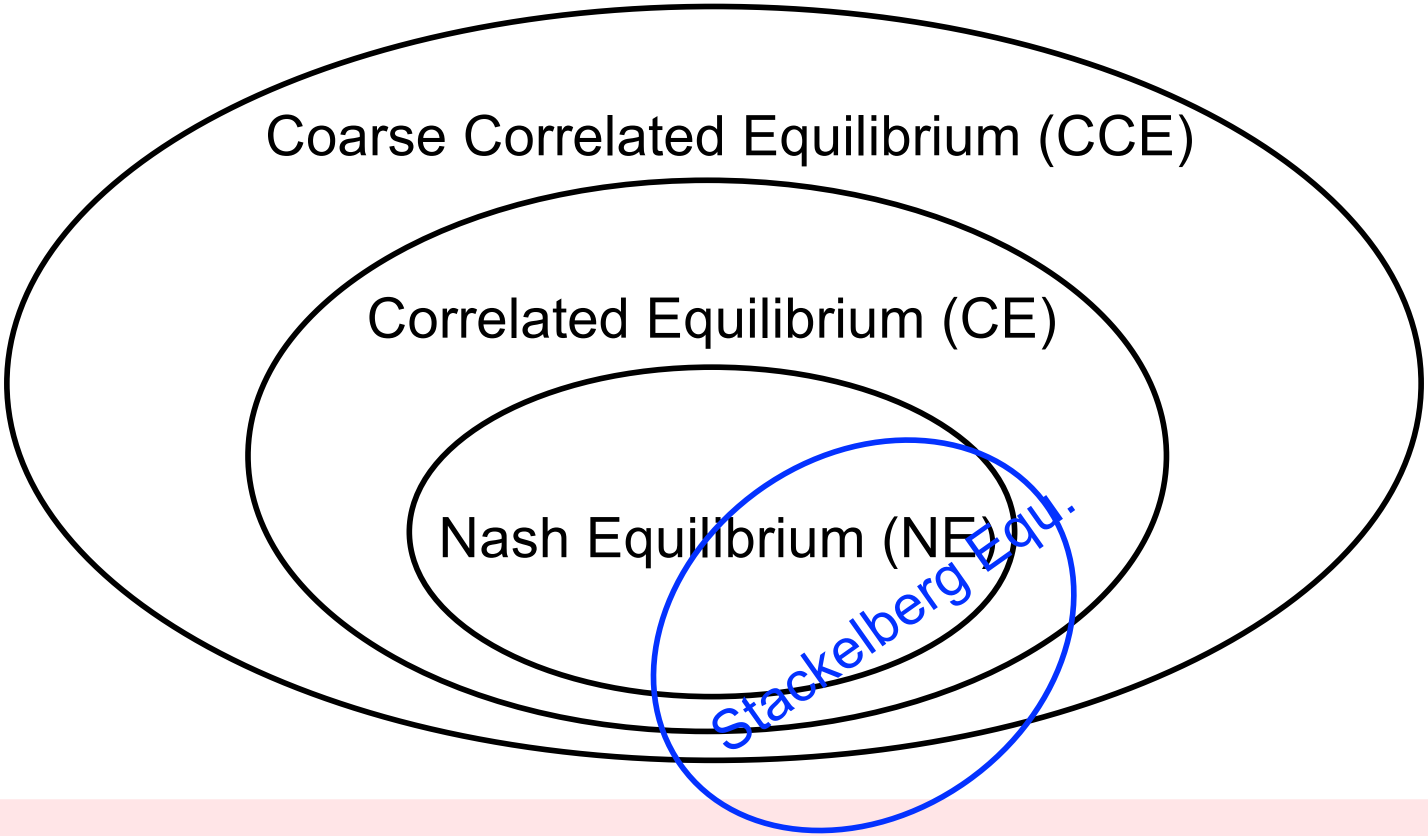
Fact. Any correlated equilibrium is a coarse correlated equilibrium.

The Equilibrium Hierarchy for Simultaneous-Move Games



There are other equilibrium concepts, but NE and CE are most often used. CCE is not used that often.

The Equilibrium Hierarchy for Simultaneous-Move Games



Where would Stackelberg equilibrium be?

- Not within any of them, somewhat different but also related
- See the paper titled "*On Stackelberg Mixed Strategies*" by Vincent Conitzer

Outline

- Correlated and Coarse Correlated Equilibrium
- Zero-Sum Games

Zero-Sum Games

- Two players: player 1 action $i \in [m] = \{1, \cdots, m\}$, player 2 action $j \in [n]$
- The game is **zero-sum** if $u_1(i,j) + u_2(i,j) = 0, \ \forall i \in [m], \ j \in [n]$
  - Models the strictly competitive scenarios
  - "Zero-sum" almost always mean "2-player zero-sum" games
  - n-player games can also be zero-sum, but not particularly interesting
- Let $u_1(x,y) = \sum_{i \in [m], j \in [n]} u_1(i,j) x_i y_j$ for any $x \in \Delta_m, y \in \Delta_n$
- (x^*, y^*) is a NE for the zero-sum game if: (1) $u_1(x^*, y^*) \geq u_1(i, y^*)$ for any $i \in [m]$; (2) $u_1(x^*, y^*) \leq u_1(x^*, j)$ for any j $\in [m]$
  - Condition $u_1(x^*, y^*) \leq u_1(x^*, j) \Leftrightarrow u_2(x^*, y^*) \geq u_2(x^*, j)$
  - We can "forget" u_2; Instead think of player 2 as minimizing player 1's utility

Maximin and Minimax Strategy

- Previous observations motivate the following definitions

Definition. $x^* \in \Delta_m$ is a maximin strategy of player 1 if it solves

$$\max_{x \in \Delta_m} \min_{j \in [n]} u_1(x, j).$$

The corresponding utility value is called maximin value of the game.

Remarks:

- x^* is player 1’s best action if he was to move first

Maximin and Minimax Strategy

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$$\max_{x \in \Delta_m} \min_{j \in [n]} u_1(x, j).$$

The corresponding utility value is called maximin value of the game.

Definition. $y^* \in \Delta_n$ is a minimax strategy of player 2 if it solves

$$\min_{y \in \Delta_n} \max_{i \in [m]} u_1(i, y).$$

The corresponding utility value is called minimax value of the game.

Remark: y^* is player 2's best action if he was to move first

Duality of Maximin and Minimax

Fact. $$\max_{x \in \Delta_m} \min_{j \in [n]} u_1(x, j) \le \min_{y \in \Delta_n} \max_{i \in [m]} u_1(i, y).$$

That is, moving first is no better.

- Let $y^* = \arg\min_{y \in \Delta_n} \max_{i \in [m]} u_1(i, y)$, so

$$\min_{y \in \Delta_n} \max_{i \in [m]} u_1(i, y) = \max_{i \in [m]} u_1(i, y^*)$$

- We have

$$\max_{x \in \Delta_m} \min_{j \in [n]} u_1(x, j) \le \max_{x \in \Delta_m} u_1(x, y^*) = \max_{i \in [m]} u_1(i, y^*)$$

Duality of Maximin and Minimax

Fact. $$\max_{x \in \Delta_m} \min_{j \in [n]} u_1(x, j) \leq \min_{y \in \Delta_n} \max_{i \in [m]} u_1(i, y).$$

Theorem. $$\max_{x \in \Delta_m} \min_{j \in [n]} u_1(x, j) = \min_{y \in \Delta_n} \max_{i \in [m]} u_1(i, y).$$

- Maximin and minimax can both be formulated as linear program

Maximin

$$\begin{aligned} \text{max} \quad & u \\ \text{s.t.} \quad & u \leq \textstyle\sum_{i=1}^{m} u_1(i,j)\, x_i, && \forall j \in [n] \\ & \textstyle\sum_{i=1}^{m} x_i = 1 \\ & x_i \geq 0, && \forall i \in [m] \end{aligned}$$

Minimax

$$\begin{aligned} \text{min} \quad & v \\ \text{s.t.} \quad & v \geq \textstyle\sum_{j=1}^{n} u_1(i,j)\, y_j, && \forall i \in [m] \\ & \textstyle\sum_{j=1}^{n} y_j = 1 \\ & y_j \geq 0, && \forall j \in [n] \end{aligned}$$

- This turns out to be primal and dual LP. Strong duality yields the equation

“Uniqueness” of Nash Equilibrium (NE)

Theorem. In 2-player zero-sum games, (x^*, y^*) is a NE if and only if x^* and y^* are the maximin and minimax strategy, respectively.

$\Leftarrow$: if x^* $[y^*]$ is the maximin [minimax] strategy, then (x^*, y^*) is a NE

- Want to prove $u_1(x^*, y^*) \geq u_1(i, y^*), \forall i \in [m]$

$$\begin{aligned} u_1(x^*, y^*) &\geq \min_{\mathrm{j}} u_1(x^*, j) \\ &= \max_{x \in \Delta_m} \min_{\mathrm{j}} u_1(x, j) \\ &= \min_{y \in \Delta_n} \max_{i \in [m]} u_1(i, y) \\ &= \max_{i \in [m]} u_1(i, y^*) \\ &\geq u_1(i, y^*), \forall i \end{aligned}$$

- Similar argument shows $u_1(x^*, y^*) \leq u_1(x^*, j), \forall j \in [n]$
- So (x^*, y^*) is a NE

“Uniqueness” of Nash Equilibrium (NE)

Theorem. In 2-player zero-sum games, (x^*, y^*) is a NE if and only if x^* and y^* are the maximin and minimax strategy, respectively.

⇒: if (x^*, y^*) is a NE, then x^* [y^*] is the maximin [minimax] strategy

- Observe the following inequalities

$$\begin{aligned} u_1(x^*, y^*) &= \max_{i \in [m]} u_1(i, y^*) \\ &\geq \min_{y \in \Delta_n} \max_{i \in [m]} u_1(i, y) \\ &= \max_{x \in \Delta_m} \min_{\mathrm{j}} u_1(x, j) \\ &\geq \min_{\mathrm{j}} u_1(x^*, j) \\ &= u_1(x^*, y^*) \end{aligned}$$

- So the two “≥” must both achieve equality.
  - The first equality implies y^* is the minimax strategy
  - The second equality implies x^* is the maximin strategy

“Uniqueness” of Nash Equilibrium (NE)

Theorem. In 2-player zero-sum games, (x^*, y^*) is a NE if and only if x^* and y^* are the maximin and minimax strategy, respectively.

Corollary.

- NE of any 2-player zero-sum game can be computed by LPs
- Players achieve the same utility in any Nash equilibrium.
  - Player 1’s NE utility always equals maximin (or minimax) value
  - This utility is also called the game value

The Collapse of Equilibrium Concepts in Zero-Sum Games

Theorem. In a 2-player zero-sum game, a player achieves the same utility in any Nash equilibrium, any correlated equilibrium, any coarse correlated equilibrium and any Strong Stackelberg equilibrium.

- Can be proved using similar proof techniques as for the previous theorem
- The problem of optimizing a player’s utility over equilibrium can also be solved easily as the equilibrium utility is the same

Thank You

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