# Homework \#3 <br> CS 6501: Learning and Game Theory (Fall'19) 

Due Tuesday 11/05 3:30 pm

General Instructions The assignment is meant to be challenging. Feel free to discuss with fellow students, however please write up your solutions independently (e.g., start writing solutions after a few hours of any discussion) and acknowledge everyone you discussed the homework with on your writeup. The course materials are all on the course website: http://www.haifeng-xu.com/cs6501fa19. You may refer to any materials covered in our class. However, any attempt to consult outside sources, on the Internet or otherwise, for solutions to any of these homework problems is not allowed.

Whenever a question asks you to "show" or "prove" a claim, please provide a formal mathematical proof. These problems have been labeled based on their difficulties. Short problems are intended to take you 5-15 minutes each and medium problems are intended to take 15-30 minutes each. Long problems may take anywhere between 30 minutes to several hours depending on whether inspiration strikes.

Finally, please write your solutions in latex - hand written solutions will not be accepted. Hope you enjoy the homework!

## Problem 1: The Posted Price Mechanism

Consider the simple setting with two buyers and $v_{1} \sim U[0,1]$ and $v_{2} \sim U[0,1]$ independently. We showed in class that for this setting, both first and second price auction achieve revenue $1 / 3$ whereas the optimal auction, which is a second-price auction with reserve $1 / 2$, achieves revenue $5 / 12$.

In this question, we will consider the posted price mechanism. That is, the seller simply posts a single and fixed price $p$ to the two buyers in sequence until one of them accepts (buyer 1 is asked first).

1. (Short, 5 points) Calculate the interim allocation and interim payment for buyer 1 and 2 (assuming $p$ is given).
2. (Medium, 5 points) Calculate the price $p$ that maximizes the revenue (among all posted price mechanisms) for the above setting.

## Problem 2: Truthfulness of VCG Mechanism

Consider the multi-item allocation problem with $m$ items and $n$ buyers. Buyer $i$ has a private value function $v_{i}(S)$ for any subset of items $S \subseteq[m]$. Recall that the Vickrey-Clarke-Groves (VCG) mechanism is the following: (1) ask each buyer to report their value function $b_{i}(S)$; (2) Compute optimal allocation $\left(S_{1}^{*}, \cdots, S_{n}^{*}\right)=\arg \max _{\left(S_{1}, \cdots, S_{n}\right)}$ is a partition of $[m] \sum_{i \in[n]} b_{i}\left(S_{i}\right) ;(3)$ Allocate $S_{i}^{*}$ to bidder $i$ and charge $i$ the following amount: $p_{i}=\max _{S_{-i} \text { is a partition of }[m]} \sum_{j \neq i} b_{j}\left(S_{j}\right)-\sum_{j \neq i} b_{j}\left(S_{j}^{*}\right)$.

1. (Short, 5 points) One special case of the above setting is when $m=1$, i.e., single-item allocation. In this case, VCG degenerates to a second-price auction. Prove that truthful bidding (i.e., bidding $b_{i}=v_{i}$ ) forms a dominant strategy equilibrium in the second-price auction.
(Note: the lecture slides actually have a formal proof for this. You may just copy it here. However, make sure you do understand that proof as you will need its idea to prove the next question.)
2. (Medium, 5 points) More generally, prove that truthful bidding (i.e., bidding $b_{i}=v_{i}$ ) forms a dominant strategy equilibrium in VCG for multiple items.

## Problem 3: The Bayesian NE for First-Price Auctions (Long, 10 points)

We proved in class that when there are two buyers with $v_{1}, v_{2} \sim U[0,1]$ independently, one (symmetric) Bayesian Nash equilibrium (BNE) is to bid $b_{i}=v_{i} / 2$ for each bidder $i=1,2$.

Now, consider running the first-price auction among $n$ buyers and $v_{i} \sim U[0,1]$ independently for $i \in[n]$. Identify a BNE for this game, and justify your answer (i.e., why is it a BNE).

## Problem 4: Randomized Reserve in Second-Price Auctions

When studying prior-independent mechanisms for i.i.d. buyers, we proved that using randomized reserve is not much worse compared to using the optimal reserve (see Lecture 13 and 14).

1. (Medium, 5 points) In Lecture 14, we use the Bulow-Klemperer (BK) theorem to prove that for $n$ i.i.d. buyers with regular distribution $F$, the revenue of directly running second-price auction (without a reserve) is at least $\left(1-\frac{1}{n}\right)$ fraction of the optimal revenue.
In this question, you are asked to use the BK theorem to prove that the revenue of running a second price auction with a random reserve $r \sim F$ is at least $\left(1-\frac{1}{n+1}\right)$ fraction of the optimal revenue.
(Hint: you may need to exploit some symmetry of this auction environment.)
2. (Medium, 5 points) The BK technique does not easily generalize to non-identical bidders. We thus described a different proof technique in proving utility guarantee for prior-independent auctions in Lecture 13. One key claim we proved is the following. Let $F$ be any regular cumulative density function over $[0, \infty)$ and define function $\widehat{R}(p)=p(1-F(p))$. Let $r^{*}=\operatorname{argmax}_{p \in \mathbb{R}} \widehat{R}(p)$. We showed that for any $t, \mathbb{E}_{r \sim F}[\widehat{R}(\max (t, r))] \geq \frac{1}{2} \widehat{R}\left(\max \left(t, r^{*}\right)\right)$.
Prove the following (stronger) claim: for any $t$ and $r^{0}>0$, we have $\mathbb{E}_{r \sim F}[\widehat{R}(\max (t, r))] \geq \frac{1}{2} \widehat{R}\left(\max \left(t, r^{0}\right)\right)$. That is, the aforementioned claim holds not only for the particular $r^{*}$, it holds for any $r^{0}>0$.
3. (Long, 10 points) Consider a setting with $2 n$ independent, but not identically distributed, bidders. For any $i=1, \cdots, n$, the values of bidder $2 i-1$ and $2 i$ are drawn from the same regular distribution $f_{i}$ independently. Now consider two different auction formats: (1) second-price auction with any personalized reserve $\vec{r}=\left(r_{1}, \cdots, r_{2 n}\right)$, denoted as $S P-P R(\vec{r})$ (see Lecture 14 for definition); (2) second-price auction with random reserve ( $S P-R R$ ) where we use bidder $(2 i-1)$ 's bid as bidder $2 i$ 's reserve and bidder $2 i$ 's bid as bidder $(2 i-1)$ 's reserve for $i=1, \cdots, n$. Prove that the expected revenue of $S P-R R$ is at least half of the revenue of $S P-P R(\vec{r})$ for any $\vec{r}$.
