Announcements

- HW2 is out, due 10/15 before class

CS6501: Topics in Learning and Game Theory
(Fall 2019)

Optimal Auction Design for Single-Item Allocation
(Part I)

Instructor: Haifeng Xu

Outline

- Mechanism Design for Single-Item Allocation
- Revelation Principle and Incentive Compatibility
- The Revenue-Optimal Auction

Single-Item Allocation

- A single and indivisible item, n buyers $\{1, \cdots, n\} = [n]$
- Buyer i has a (private) value $v_i \in V_i$ about the item
- Outcome: choice of the winner of the item, and payment p_i from each buyer i
- Objectives: maximize revenue
  - Last lecture: VCG auction maximizes welfare even for multiple items

The Mechanism Design Problem

Mechanism Design for Single-Item Allocation

Described by $\langle n, V, X, u \rangle$ where:

- $[n] = \{1, \cdots, n\}$ is the set of n buyers
- $V = V_1 \times \cdots \times V_n$ is the set of all possible value profiles
- $X = \{e_0, e_1, \cdots, e_n\}$ is the set of all possible allocation outcomes
- $u = (u_1, \cdots, u_n)$ where $u_i = v_i x_i - p_i$ is the utility function of i for any outcome $x \in X$ and payment p_i required from i

Objective: maximize revenue $\sum_{i \in [n]} p_i$

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Objective: maximize revenue $\sum_{i \in [n]} p_i$

- Cannot have any guarantee without additional assumptions
- Will assume public prior knowledge on buyer values. For convenience, think of $v_i \sim f_i$ independently
  - Most results of this lecture hold for correlated v_i's, but easier to think for independent cases

The Mechanism Design Problem

Mechanism Design for Single-Item Allocation

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Remarks:

- General mechanism design problem can be defined similarly
- $u_i = v_i x_i - p_i$ is called quasi-linear utility function
  - Not the only form of utility functions, but widely adopted
- Typically, $V_1 = \mathbb{R}_+$, but can also be intervals like $[a, b]$

The Mechanism Design Problem

Mechanism Design for Single-Item Allocation

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Remarks:

- Assume risk neural players – i.e., all players maximize expected utilities
- Will guarantee $\mathbb{E}[u_i] \geq 0$ (a.k.a., individually rational or IR)
  - Otherwise, players would not even bother coming to your auction

The Design Space – Mechanisms

A mechanism (i.e., the game) is specified by $\langle A, g \rangle$ where:

- $\mathrm{A} = A_1 \times \cdots \times A_n$ where A_i is allowable actions for buyer i
- $g: A \to [x, p]$ maps an action profile to [an allocation outcome $x(a)$ + a vector of payments $p(a)$] for any $a = (a_1, \cdots, a_n) \in A$

- That is, we will design $\langle A, g \rangle$

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- That is, we will design $\langle A, g \rangle$
- Players' utility function will be fully determined by $\langle A, g \rangle$
- This is a game with incomplete information – v_i is privately known to player i; all other players only know its prior distribution

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Example 1: first-price auction

- $A_i = \mathbb{R}_+$ for all i
- $g(a)$ allocates the item to the buyer $i^* = \arg\max_{i \in [n]} a_i$ and asks i^* to pay a_{i^*}, and all other buyers pay 0

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Example 2: second-price auction

- $A_i = \mathbb{R}_+$ for all i
- $g(a)$ allocates the item to the buyer $i^* = \arg\max_{i \in [n]} a_i$ and asks i^* to pay $\max 2_i \, a_i$, and all other buyers pay 0

The Design Space – Mechanisms

A mechanism (i.e., the game) is specified by $\langle A, g \rangle$ where:

- $\mathrm{A} = A_1 \times \cdots \times A_n$ where A_i is allowable actions for buyer i
- $g: A \rightarrow [x, p]$ maps an action profile to [an allocation outcome $x(a)$ + a vector of payments $p(a)$] for any $a = (a_1, \cdots, a_n) \in A$

- In general, A, g can be really arbitrary, up to your design
- E.g, the following is a valid – though bad – mechanism
  - $A_i = \{jump\ twice\ (J),\ look\ 45^{\circ}\ up\ (L)\}$
  - $x(a)$ gives the item to anyone of L uniformly at random
  - $p(a)$ asks everyone to pay $\$0$

Revenue at Equilibrium

- How to predict/estimate how much revenue we achieve?
- Revenue = expected revenue at (Bayesian) Nash equilibrium
- Due to incomplete information, player i's strategy is $s_i: V_i \to \Delta(A_i)$ where $s_i(v_i)$ is the mixed strategy of i with private value v_i
- Expected utility of i with value v_i in mechanism $\langle A, g \rangle$ is

$$\mathbb{E}_{(a_i, a_{-i}) \sim (s_i(v_i), s_{-i}(v_{-i}))}[v_i x_i(a_i, a_{-i}) - p_i(a_i, a_{-i})]$$

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$$= U_i \; (\, s_i(v_i) \; | v_i, s_{-i})$$

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$$= U_i \left(s_i(v_i) \,|v_i, s_{-i} \right)$$

Strategy profile $s^* = (s_1^*, \cdots, s_n^*)$ is a Bayes Nash Equilibrium (BNE) for mechanism $\langle A, g \rangle$ if for any player i and value v_i

$$U_i(s_i^*(v_i) \,|v_i, s_{-i}^*) \geq U_i(a_i|v_i, s_{-i}^*), \qquad \forall\, a_i \in A_i$$

That is, $s_i^*(v_i)$ is a best response to s_{-i}^* for any i and v_i.

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Theorem. Any finite Bayesian game admits a mixed BNE.

- Can be proved by Nash's theorem
- It so happens that in many natural Bayesian games we look at, there will be a pure BNE

Revenue at Equilibrium

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That is, $s_i^*(v_i)$ is a best response to s_{-i}^* for any i and v_i.

Q: what is the BNE for second-price auction?

Truthful bidding is a dominant strategy equilibrium (thus also BNE)

- Truthful bidding is a dominant strategy. That is, for any i and v_i, for any a_{-i}, we have

$$v_i x_i(v_i, a_{-i}) - p_i(v_i, a_{-i}) \geq v_i x_i(a_i', a_{-i}) - p_i(a_i', a_{-i})$$

- Bidding v_i remains optimal after expectation over a_{-i} and v_{-i}

BNE for First-Price Auction

- In general, still an open question in economics and CS
- Can be computed for simple cases

BNE for First-Price Auction

Example: Two bidders, $v_1, v_2 \sim U([0,1])$ independently

Claim. $b_i(v_i) = v_i/2$ forms a Bayes Nash Equilibrium.

Proof

- By symmetry, w.l.o.g., focus on bidder 1
- Assume bidder 2 uses $b_2 = v_2/2$; $\mathbb{P}(b_2 \le b) = \min(2b, 1)$, $\forall b \in [0,1]$
- Utility of bidder 1 with value v_1 and any bid b_1 is

$$\mathbb{P}[b_1 \ge b_2)] \times (v_1 - b_1)$$
$$= \min(2b_1, 1) \times (v_1 - b_1)$$

- Which b_1 maximizes this utility?
  - If $b_1 \ge 1/2$, it decreases in b_1, so should bid at most $1/2$
  - Thus, utility is $2b_1(v_1 - b_1)$, which is maximized at $b_1 = v_1/2$

The Main Points . . .

- A mechanism $\langle A, g \rangle$ specifies action space A and a mapping from action profiles to [an allocation outcome + payments]
- Any mechanism describes a Bayesian game
- We compute the revenue at some Bayes Nash equilibrium
  - Since this is what we predict the players will behave
  - Will design mechanisms that are very easy for players to play

Optimal Mechanism Design

Design mechanism $\langle A, g \rangle$ to maximize revenue at the BNE

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Design mechanism $\langle A, g\rangle$ to maximize revenue at the BNE

First major challenge: with so many possible actions in this world, what should I use?

- Revelation principle says that you only need them to report their value v_i

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- Mechanism Design for Single-Item Allocation
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Direct Revelation Mechanisms

Definition. A mechanism $\langle A, g \rangle$ is a direct revelation mechanism if $A_i = V_i$ for all i. In this case, the mechanism is described by g.

- That is, the action for each player is to "report" their value (but they don't have to be honest…yet)
- Examples: second-price auction, first-price auction
- Note: this restriction limits our design space as it limits our choice of A_i's
  - Not clear yet whether this restriction will reduce our best achievable revenue
  - Will show that it indeed does not!

Incentive-Compatibility

Definition. A direct revelation mechanism g is Bayesian incentive-compatible (a.k.a., truthful or BIC) if truthful bidding forms a Bayes Nash equilibrium in the resulting game

- A similar but stronger IC requirement

Definition. A direct revelation mechanism g is Dominant-Strategy incentive-compatible (a.k.a., truthful or DIC) if truthful bidding is a dominant-strategy equilibrium in the resulting game

- A DIC mechanism is also BIC

Incentive-Compatibility: Examples

Second-price auction is dominant-strategy incentive-compatible, and thus also Bayesian incentive-compatible.

First-price auction is not Bayesian incentive-compatible.

Incentive-Compatibility: Examples

Second-price auction is dominant-strategy incentive-compatible, and thus also Bayesian incentive-compatible.

First-price auction is not Bayesian incentive-compatible.

Definition (Posted price). The auctioneer simply posts a fixed price p to players in sequence until one buyer accepts.

- Not exactly a direct revelation mechanism as buyer only chooses to accept or not accept, while not report their value
- But can be trivially modified to a direct revelation mechanism by asking buyers to report their value and $v_i \geq p$ leads to an accept
- Both DIC and BIC

Incentive-Compatibility: Examples

- Consider the following mechanism for the case with two bidders and $v_1, v_2 \sim U([0,1])$ independently

Modified First-Price Auction. Solicit bid b_1, b_2; highest bid wins and pays half its bid, i.e., $\max(b_1, b_2)/2$.

- Equivalently, simulate first price auction where bidders bid $b_1/2, b_2/2$

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- Equivalently, simulate first price auction where bidders bid $b_1/2, b_2/2$

Claim. Modified first-price auction is BIC in *the above example*

- Assuming bidder 2 truthfully bids v_2. This is as if bidder 1 faces a first price auction where bidder 2 bid $b_2 = v_2/2$ and his bid is $b_1 = b/2$ if he bids b in the modified version
- Since $b_i(v_i) = v_i/2$ is a BNE of the first-price auction, thus $b/2 = v_i/2$ (i.e., $b = v_i$) must be a best response

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- Equivalently, simulate first price auction where bidders bid $b_1/2, b_2/2$

Claim. Modified first-price auction is BIC in *the above example*

Key insights:

- Whatever manipulations bidders do at equilibrium, the auctioneer can directly implement it on behalf of the bidders, thus in the modified mechanism being truthful becomes optimal for bidders
- This ideas turns out to generalize

The Revelation Principle

Theorem. If there is a mechanism that achieves revenue R at a Bayes Nash equilibrium [resp. dominant-strategy equilibrium], then there is a direct revelation, Bayesian incentive-compatible [resp. DIC] mechanism achieving revenue R.

Remarks

- Can be stated more generally, but this version is sufficient for our purpose of optimal auction design
  - The same proof idea
- Can thus focus on BIC mechanisms henceforth; Often omit word "direction revelation" as we almost always design DR mechanisms

The Revelation Principle

Theorem. If there is a mechanism that achieves revenue R at a Bayes Nash equilibrium [resp. dominant-strategy equilibrium], then there is a direct revelation, Bayesian incentive-compatible [resp. DIC] mechanism achieving revenue R.

This simplifies our mechanism design task

Optimal Mechanism Design for Single-Item Allocation

Given instance $\langle n, V, X, u \rangle$, supplemented with prior $\{f_i\}_{i \in [n]}$, design the allocation function $x: V \to X$ and payment $p: V \to \mathbb{R}^n$ such that truthful bidding is a BNE in the following Bayesian game:

1. Solicit bid $b_1 \in V_1, \cdots, b_n \in V_n$
2. Select allocation $x(b_1, \cdots, b_n) \in X$ and payment $p(b_1, \cdots, b_n)$

Design goal: maximize expected revenue

Proof (Bayesian Setting)

- Consider any mechanism $\langle A, g\rangle$ with BNE strategies $s_i: V_i \rightarrow A_i$
- Define a new mechanism that simulates the BNE on behalf of players

Modified Mechanism.

1. Solicit reported value (as bid) $b_1 \in V_1, \cdots, b_n \in V_n$
2. Choose allocation outcome $\bar{g}(b_1, \cdots, b_n) = g(s_1(b_1), \cdots, s_n(b_n))$ and payment vector $\bar{p}(b_1, \cdots, b_n) = p(s_1(b_1), \cdots, s_n(b_n))$
   - (If s_i's are mixed strategies, add expectation signs)

Argue that truthful bidding is a BNE in the modified mechanism

- Focus on i with value v_i, and assume all other bidders bid truthfully
- This is as if all other bidders play $s_{-i}(v_{-i})$ in original mechanism
- Then, $s_i(v_i)$ must be bidder i'th optimal bid by definition of BNE
- Since auctioneer will apply function s_i to i's bid in the modified mechanism, he should just bid v_i

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Optimal (Bayesian) Mechanism Design

- Previous formulation and simplification leads to the following optimization problem

$$\max_{x,p} \ \mathbb{E}_{v\sim f} \ \sum_{i=1}^{n} p_i(v_1, \cdots, v_n)$$

$$\begin{aligned}
\text{s.t.} \quad & \mathbb{E}_{v_{-i}\sim f_{-i}}\left[v_i x_i(v_i, v_{-i}) - p_i(v_i, v_{-i})\right] \\
& \quad \geq \mathbb{E}_{v_{-i}\sim f_{-i}}\left[v_i x_i(b_i, v_{-i}) - p_i(b_i, v_{-i})\right], && \forall i \in [n], v_i, b_i \in V_i \\
& \mathbb{E}_{v_{-i}\sim f_{-i}}\left[v_i x_i(v_i, v_{-i}) - p_i(v_i, v_{-i})\right] \geq 0, && \forall i \in [n], v_i \in V_i \\
& x(v) \in X, && \forall v \in V
\end{aligned}$$

Optimal (Bayesian) Mechanism Design

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BIC constraints

$$\max_{x,p} \ \mathbb{E}_{v\sim f} \ \textstyle\sum_{i=1}^{n} p_i(v_1, \cdots, v_n)$$

$$\text{s.t.} \quad \boxed{\begin{aligned} &\mathbb{E}_{v_{-i}\sim f_{-i}}\left[v_i x_i(v_i, v_{-i}) - p_i(v_i, v_{-i})\right] \\ &\quad \geq \mathbb{E}_{v_{-i}\sim f_{-i}}\left[v_i x_i(b_i, v_{-i}) - p_i(b_i, v_{-i})\right], \end{aligned}} \qquad \forall i \in [n], v_i, b_i \in V_i$$

$$\boxed{\mathbb{E}_{v_{-i}\sim f_{-i}}\left[v_i x_i(v_i, v_{-i}) - p_i(v_i, v_{-i})\right] \geq 0,} \qquad \forall i \in [n], v_i \in V_i$$

$$x(v) \in X, \qquad \forall v \in V$$

Individually rational (IR) constraints

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$$\max_{x,p} \; \mathbb{E}_{v\sim f} \; \sum_{i=1}^{n} p_i(v_1, \cdots, v_n)$$

$$\begin{aligned} \text{s.t.} \quad & \mathbb{E}_{v_{-i}\sim f_{-i}} [v_i x_i(v_i, v_{-i}) - p_i(v_i, v_{-i})] \\ & \quad \geq \mathbb{E}_{v_{-i}\sim f_{-i}} [v_i x_i(b_i, v_{-i}) - p_i(b_i, v_{-i})], && \forall i \in [n], v_i, b_i \in V_i \\ & \mathbb{E}_{v_{-i}\sim f_{-i}} [v_i x_i(v_i, v_{-i}) - p_i(v_i, v_{-i})] \geq 0, && \forall i \in [n], v_i \in V_i \\ & x(v) \in X, && \forall v \in V \end{aligned}$$

- This problem is challenging because we are optimizing over functions $x: V \to X$ and $p: V \to \mathbb{R}^n$

Optimal DIC Mechanism Design

- Designing optimal dominant-strategy incentive compatible (**DIC**) mechanism is a strictly more constrained optimization problem

$$\max_{x,p} \; \mathbb{E}_{v\sim f} \sum_{i=1}^{n} p_i(v_1, \cdots, v_n)$$

$$\begin{aligned} \text{s.t.} \quad & \mathbb{E}_{v_{-i}\sim f_{-i}} [v_i x_i(v_i, v_{-i}) - p_i(v_i, v_{-i})] \\ & \quad \geq \mathbb{E}_{v_{-i}\sim f_{-i}} [v_i x_i(b_i, v_{-i}) - p_i(b_i, v_{-i})], & \forall i \in [n], v_i, b_i \in V_i \\ & \mathbb{E}_{v_{-i}\sim f_{-i}} [v_i x_i(v_i, v_{-i}) - p_i(v_i, v_{-i})] \geq 0, & \forall i \in [n], v_i \in V_i \\ & x(v) \in X, & \forall v \in V \end{aligned}$$

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$$\text{s.t.} \quad [v_i x_i(v_i, v_{-i}) - p_i(v_i, v_{-i})] \qquad \forall\, v_{-i}$$

$$\geq \quad [v_i x_i(b_i, v_{-i}) - p_i(b_i, v_{-i})], \qquad \forall i \in [n], v_i, b_i \in V_i$$

$$\mathbb{E}_{v_{-i}\sim f_{-i}}[v_i x_i(v_i, v_{-i}) - p_i(v_i, v_{-i})] \geq 0, \qquad \forall i \in [n], v_i \in V_i$$

$$x(v) \in X, \qquad \forall v \in V$$

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$$
\begin{aligned}
\max_{x,p} \quad & \mathbb{E}_{v\sim f} \sum_{i=1}^{n} p_i(v_1, \cdots, v_n) \\
\text{s.t.} \quad & [v_i x_i(v_i, v_{-i}) - p_i(v_i, v_{-i})] & \forall\, v_{-i} \\
& \geq \quad [v_i x_i(b_i, v_{-i}) - p_i(b_i, v_{-i})], & \forall i \in [n], v_i, b_i \in V_i \\
& \mathbb{E}_{v_{-i}\sim f_{-i}}[v_i x_i(v_i, v_{-i}) - p_i(v_i, v_{-i})] \geq 0, & \forall i \in [n], v_i \in V_i \\
& x(v) \in X, & \forall v \in V
\end{aligned}
$$

Corollary. Optimal DIC mechanism achieves revenue at most that of optimal BIC mechanism.

Myerson's Optimal Auction

Theorem (informal). For single-item allocation with prior distribution $v_i \sim f_i$ independently, the following auction is BIC and optimal:

1. Solicit buyer values $v_1, \cdots, v_n$
2. Transform v_i to "virtual value" $\phi_i(v_i)$ where $\phi_i(v_i) = v_i - \frac{1-F_i(v_i)}{f_i(v_i)}$
3. If there exists $\phi_i(v_i) \geq 0$, allocate item to $i^* = \arg\max_{i \in [n]} \phi_i(v_i)$ and charge him the minimum bid needed to win, i.e., $\phi_i^{-1}\left(\max\left(\max_{j \neq i^*} \phi_j(v_j), 0\right) \right)$; Other bidders pay 0.
4. If $\phi_i(v_i) < 0$ for all i, keep the item and no payments

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4. If $\phi_i(v_i) < 0$ for all i, keep the item and no payments

- Recall second-price auction, we also charge the minimum bid to win, but directly use the bid to determine winner
- Key differences from second-price auction: (1) use virtual value to determine winner; (2) added a "fake bidder" with virtual value 0

Remarks

Myerson's optimal auction is noteworthy for many reasons

- Matches practical experience: when buyer values are i.i.d, optimal auction is a second price auction with reserve $\phi^{-1}(0)$.
- Applies to "single parameter" problems more generally
- The optimal BIC mechanism just so happens to be DIC and deterministic!!
  - Not true for multiple items – there exists revenue gap even when selling two items to two bidders

Thank You

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