# Announcements

- Collect HW1 grading (see Collab for sample solution)
- HW 2 is due next Tuesday
  - No class on next Tuesday, but TAs will be here to collect HW
- HW 3 will be out by the end of this week
  - Likely will have a very light HW 4 or no HW 4
- Instructions for course project will be out by the end of this week

# CS6501: Topics in Learning and Game Theory
# (Fall 2019)

## Simple Auctions

Instructor: Haifeng Xu

# Outline

- Prior-Independent Auctions for *I.I.D. Buyers*
- Intricacy of Optimal Auction for *Independent Buyers*
- Simple Auction for *Independent Buyers*

# IID Buyers: What Have We Learned So Far?

- Optimal auction is a second-price auction with reserve $\phi^{-1}(0)$
  - Notation: buyer value $v_i \sim f$ (regular) and $\phi(v) = v - \frac{1-F(v)}{f(v)}$
- Optimal auction (unrealistically) requires completely knowing $f$

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- Optimal auction (unrealistically) requires completely knowing $f$
- Last lecture – prior-independent auction
  - Still assume $v_i \sim f$, but do not know $f$
  - Guarantee roughly $1/2$ of the optimal revenue for any $n \geq 2$
  - Like ML: data drawn from unknown distributions

Second-Price auction with Random Reserve (SP-RR)

1. Solicit buyer values $v_1, \cdots, v_n$
2. Pick $j \in [n]$ uniformly at random as the reserve buyer
3. Run second-price auction with reserve $v_j$ but only among bidders in $[n] \setminus \{j\}$.

# IID Buyers: What Have We Learned So Far?

Key insights from the proof of ½ approximation:

- Discarding a buyer does not hurt revenue much

**Lemma 1.** The expected optimal revenue for an environment with $(n-1)$ buyers is at least $\frac{n-1}{n}$ fraction of the optimal expected revenue for $n$ buyers.

# IID Buyers: What Have We Learned So Far?

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**Lemma 1.** The expected optimal revenue for an environment with $(n-1)$ buyers is at least $\frac{n-1}{n}$ fraction of the optimal expected revenue for $n$ buyers.

- Using random reserve is not bad
  - SP-OR: second price auction with optimal reserve $r^* = \phi^{-1}(0)$
  - SP-RR: second price auction with random reserve $r \sim F$

**Lemma 2.** Rev(SP-RR) $\geq \frac{1}{2}$ Rev(SP-OR) for any $n \geq 1$ and regular $F$.

# IID Buyers: What Have We Learned So Far?

Next, we show that even directly running second-price auction without reserve is not bad for i.i.d. buyers

- Built upon a fundamental result by [Bulow-Klemperer, '96]
- Can be used to strengthen previous approximation guarantee
  - Drawback: this technique does not easily generalize to independent buyers

# IID Buyers: What Have We Learned So Far?

Next, we show that even directly running second-price auction without reserve is not bad for i.i.d. buyers

- Built upon a fundamental result by [Bulow-Klemperer, '96]
- Can be used to strengthen previous approximation guarantee
  - Drawback: this technique does not easily generalize to independent buyers
- Inspired the whole research agenda on simple yet approximately optimal auction design

Note: "Simple" is a subjective judge, no formal definition

# The Bulow-Klemperer Theorem

**Theorem.** For any $n(\geq 1)$ i.i.d. buyers with regular $F$, we have

$$Rev_{n+1}(SP) \geq Rev_n(SP\text{-}OR)$$

Notations

- SP – second-price auction;
- $Rev_n(M)$ – revenue of any mechanism $M$ for $n$ i.i.d buyers

# The Bulow-Klemperer Theorem

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- That is, second-price auction with an additional buyer achieves higher revenue than the optimal auction
- Insight: more competition is better than finding the right auction format

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Proof: an application of Myerson's Lemma

**Lemma.** Consider any BIC mechanism $M$ with interim allocation $x$ and interim payment $p$, normalized to $p_i(0) = 0$. The expected revenue of $M$ is equal to the expected virtual welfare served

$$\sum_{i=1}^{n} \mathbb{E}_{v_i \sim f_i}[\, \phi_i(v_i) x_i(v_i)]$$

# The Bulow-Klemperer Theorem

**Theorem.** For any $n(\geq 1)$ i.i.d. buyers with regular $F$, we have

$$Rev_{n+1}(SP) \geq Rev_n(SP\text{-}OR)$$

Proof: an application of Myerson's Lemma

- Consider the following auction for $n + 1$ buyers:
  1. Run SP-OR for first $n$ buyers;
  2. If not sold, give the item to bidder $n + 1$ for free
- Two observations
  a. This auction always allocates the item, and is BIC
  b. Achieves the same revenue as $Rev_n(SP\text{-}OR)$
- We argue that SP for $n + 1$ buyers achieves higher revenue

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**Claim.** SP has highest revenue among auctions that always allocate item

✓ Myerson's lemma: revenue = virtual welfare served

✓ SP always gives the item to the one with highest virtual welfare

# The Bulow-Klemperer Theorem

**Theorem.** For any $n(\geq 1)$ i.i.d. buyers with regular $F$, we have

$$Rev_{n+1}(SP) \geq Rev_n(SP\text{-}OR)$$

**Corollary.** For any $n \geq 2$, $Rev_n(SP) \geq (1 - \frac{1}{n}) Rev_n(SP\text{-}OR)$

Remarks:

- SP is prior-independent, simple and approximately optimal
- Recovers previous result when $n = 2$
  - With even better guarantee when $n \geq 3$

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Proof:

$$Rev_n(SP) \geq Rev_{n-1}(SP\text{-}OR)$$

$$\geq (1 - \frac{1}{n}) Rev_n(SP\text{-}OR)$$

Since discarding a bidder does not hurt revenue much

# Outline

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# Optimal Auction for Independent Buyers

**Theorem.** For single-item allocation with regular value distribution $v_i \sim f_i$ independently, the following auction is BIC and optimal:

1. Solicit buyer values $v_1, \cdots, v_n$
2. Transform $v_i$ to "virtual value" $\phi_i(v_i)$ where $\phi_i(v_i) = v_i - \frac{1-F_i(v_i)}{f_i(v_i)}$
3. If $\phi_i(v_i) < 0$ for all $i$, keep the item and no payments
4. Otherwise, allocate item to $i^* = \arg\max_{i \in [n]} \phi_i(v_i)$ and charge him the minimum bid needed to win, i.e., $\phi_i^{-1}\left( \max\left(\max_{j \neq i^*} \phi_j(v_j), 0\right) \right)$.

# An Example

- Two bidders, $v_1 \sim U[0,1]$, $v_2 \sim U[0,100]$
- $\phi_1(v_1) = v_1 - \frac{1-F_1(v_1)}{f_1(v_1)} = 2v_1 - 1$, $\phi_2(v_2) = 2v_2 - 100$

Optimal auction has the following rules:

- ✓ When $v_1 > ½, v_2 < 50$, allocate to bidder $1$ and charge ½
- ✓ When $v_1 < ½, v_2 > 50$, allocate to bidder $2$ and charge $50$
- ✓ When $0 < 2v_1 - 1 < 2v_2 - 100$, allocate to bidder $2$ and charge $(99 + 2v_1)/2$ (a tiny bit above $50$)
- ✓ When $0 < 2v_2 - 100 < 2v_1 - 1$, allocate to bidder $1$ and charge $(2v_2 - 99)/2$ (a tiny bit above $1/2$)

- Roughly, want to give it to bidder 2 for 50, and otherwise give it to bidder 1 for 0.5
- Optimal auction is less natural, especially with many buyers

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**Q:** Is there a simple auction that's approximately optimal?

Note: second-price auction alone does not work → The above example

# Outline

- Prior-Independent Auctions for *I.I.D. Buyers*

- Intricacy of Optimal Auction for *Independent Buyers*

- Simple Auction for *Independent Buyers*
  - Notations: $v_i \sim f_i$ for $i \in [n]$

# Simple Auctions are Approximately Optimal

- Second-price auction with a single reserve also achieves $\approx 1/4$ fraction of OPT
  - The best reserve will depend on $f_i$'s
- Second-price auction with personalized reserve (depending on the priors) achieves $\approx 1/2$ fraction of OPT
  - Again, reserves will depend on $f_i$'s

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Next: will prove this result

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  - Proof is based on an elegant result from optimal stopping theory

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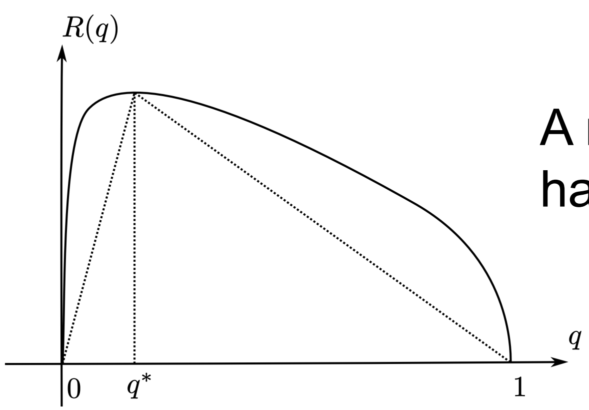
  - The best reserve will depend on $f_i$'s
- Second-price auction with personalized reserve (depending on the priors) achieves $\approx 1/2$ fraction of OPT
  - Proof is based on an elegant result from optimal stopping theory
  - Dependence on prior can be resolved using similar ideas from last lecture, with an additional loss of approximation factor $1/2$




A random reserve extracts at least half of any deterministic revenue

# Second-Price Auction with Personalized Reserves

Second-Price Auction with Personalized Reserves (SP-PR)

**Parameters**: $r_1, r_2, \cdots, r_n$

1. Solicit values $v_1, \cdots, v_n$
2. Select potential buyer set $S = \{i\colon\ v_i \geq r_i\}$
3. If $S = \emptyset$, keep the item; Otherwise, allocate to $i^* = \arg\max_{i \in S} v_i$ and charges him $\max(\max2_{i \in S} v_i\ ,\ r_{i^*})$

- Note: reserves are chosen before values are solicited

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- Note: reserves are chosen before values are solicited
- Example
  - Two bidders, $r_1 = 0.5$, $r_2 = 50$

**Q1:** if $v_1 = 0.6$, $v_2 = 49$, what is the outcome?

**Q2:** if $v_1 = 0.6$, $v_2 = 51$, what is the outcome?

# Second-Price Auction with Personalized Reserves

Second-Price Auction with Personalized Reserves (SP-PR)

**Parameters**: $r_1, r_2, \cdots, r_n$

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2. Select potential buyer set $S = \{i \colon v_i \geq r_i\}$
3. If $S = \emptyset$, keep the item; Otherwise, allocate to $i^* = \arg\max_{i \in S} v_i$ and charges him $\max(\max2_{i \in S} v_i \; , \; r_{i^*})$

**Claim.** SP-PR is dominant-strategy incentive compatible.

**Theorem.** There exists a $\theta$ such that the SP-PR with reserves $\phi_1^{-1}(\theta), \cdots, \phi_n^{-1}(\theta)$ achieves revenue at least ½ of OPT.

Remarks:

- $\theta$ can be efficiently computed, but depends on $f_i$'s
- $\phi_1^{-1}(\theta), \cdots, \phi_n^{-1}(\theta)$ are just one choice of reserves, not necessarily optimal – nevertheless, enough to guarantee ½ of OPT
- To prove this theorem, we take a small detour to a relevant problem from optimal stopping theory

# The Jewelry Selection Game

$R_1 \sim g_1$ $\quad$ $R_2 \sim g_2$ $\quad \cdots\cdots \quad$ $R_n \sim g_n$

$g_i$'s publicly known

- You open boxes sequentially from $1, \cdots, n$
- After open $i$, you observe realized jewelry reward $R_i$ and decides to: either (1) accept $R_i$ and stop; or (2) give up $R_i$ and continue

**Question:** Is there a strategy for playing the game, whose expected reward competes with that of a prophet who sees realized $R_1, \cdots, R_n$?

The prophet will get $\mathbb{E}_{R_i \sim g_i}[\max_{i \in [n]} R_i]$

# The Jewelry Selection Game

$R_1 \sim g_1$ $R_2 \sim g_2$ $R_n \sim g_n$

$g_i$'s publicly known

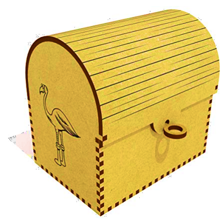
- A strategy is a stopping rule, i.e., deciding a time $\tau$ to stop

A natural class of strategies is threshold strategy, parameterized by $\theta$: pick the first $R_i \geq \theta$

$\theta$ has to be carefully chosen beforehand

- Too large: ends up picking nothing (or pick $R_n$)
- Too small: lose the change of picking a large reward

# The Jewelry Selection Game



$R_1 \sim g_1$

$R_2 \sim g_2$

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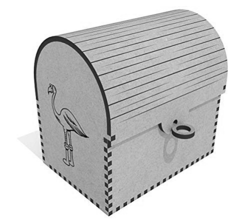

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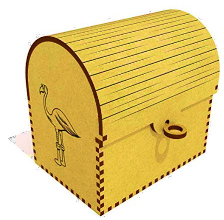
$g_i$'s publicly known

- A strategy is a stopping rule, i.e., deciding a time $\tau$ to stop

A natural class of strategies is threshold strategy, parameterized by $\theta$: pick the first $R_i \geq \theta$

Note: after $\theta$ is chosen, the stop time $\tau$ depends on randomness of $R_1, \cdots, R_n$

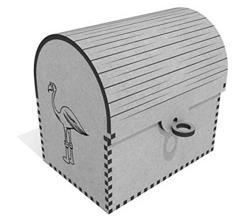
# The Jewelry Selection Game



$R_1 \sim g_1$

$R_2 \sim g_2$

. . . .



$R_n \sim g_n$

$g_i$'s publicly known

**Theorem** [Prophet Inequality]**.** There exists a $\theta$ such that the stopping time $\tau$ determined by threshold strategy $\theta$ satisfies

$$\mathbb{E}[R_\tau] \geq \frac{1}{2}\mathbb{E}[\max_{i \in [n]} R_i].$$

- $\theta$ depends on $g_i$'s but not $R_i$'s
- Both expectations are over randomness of $R_i$'s

# Back to Our Auction Problem…

**Theorem.** There exists a $\theta$ such that the SP-PR with reserves $\phi_1^{-1}(\theta), \cdots, \phi_n^{-1}(\theta)$ achieves revenue at least ½ of OPT.

Proof:

- Optimal auction picks the largest among $\phi_1(v_1), \cdots, \phi_n(v_n), 0$
  - Like the prophet
- By previous theorem, there exists a $\theta$ such that if we allocate to any $i$ with $\phi_i(v_i) \geq \theta$, the collected virtual welfare (and thus revenue) will be at least half of the optimal
  - Equivalently, allocate to any $i$ with $v_i \geq \phi_i^{-1}(\theta) = r_i$
- SP-PR uses just a particular way to pick such an $i$

# Proof of Prophet Inequality

- See reading materials

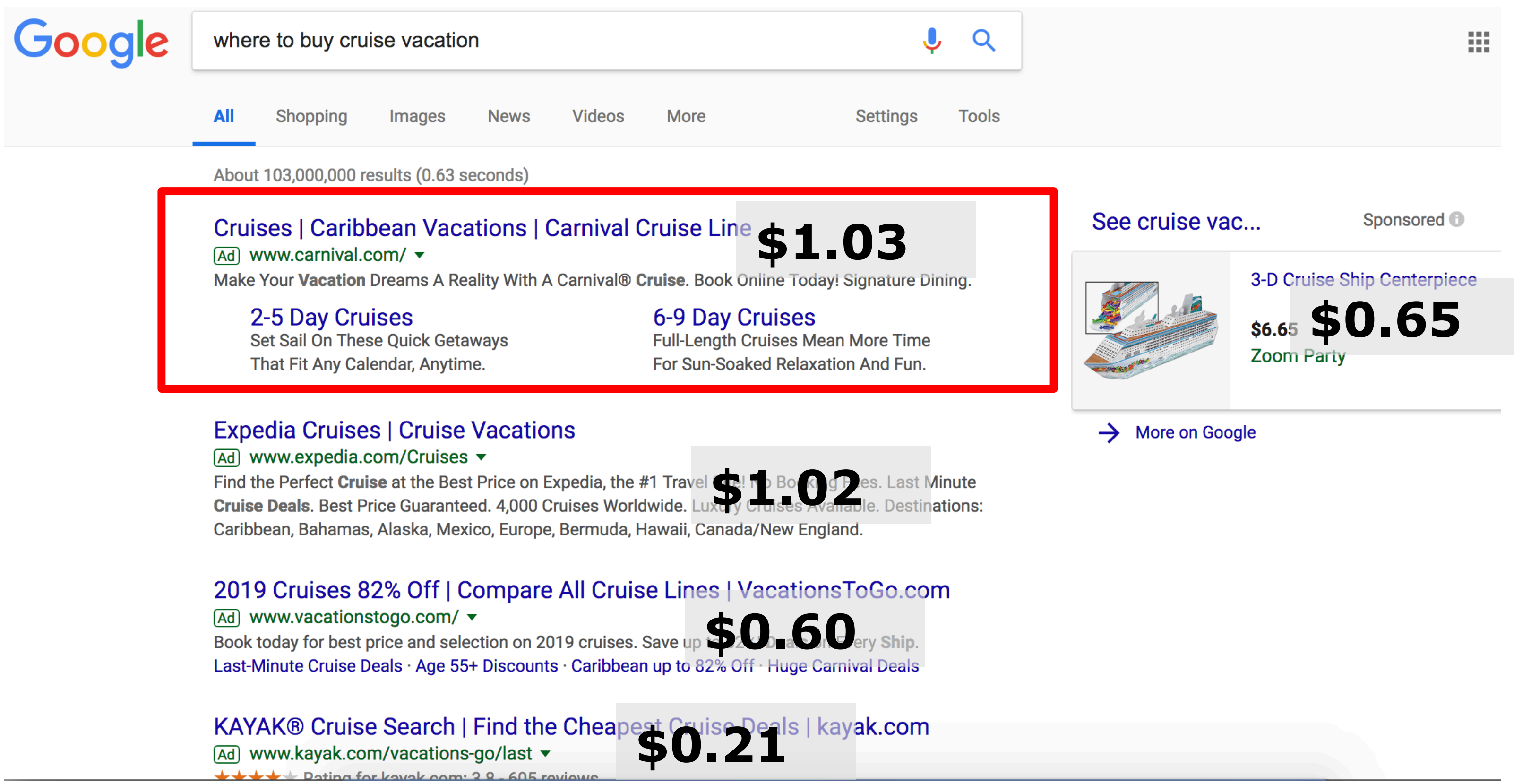
# Concluding Remarks

- $\theta$ depends on prior distributions
  - Can be resolved by using randomized reserve from the "reserve bidder", but will lose an additional factor ½
  - Need certain non-singularity assumption
- Design of simple approximately optimal auctions is still a hot topic in mechanism design, particularly for selling multiple products
  - Exactly optimal auction is extremely difficult, has been open for many years, and has many weird performances
  - Simple auctions with performance guarantee helps to identify crucial factors for practitioners

# Concluding Remarks

➢ Examples of (simple) auctions in practice, where CS studies have made impact

Ad Auctions: billions of dollars of revenue each year



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- Examples of (simple) auctions in practice, where CS studies have made impact

**Spectrum Auctions**: sell spectrum licenses to network operators

### FCC launches first U.S. high-band 5G spectrum auction

David Shepardson

3 MIN READ

(Reuters) - The Federal Communications Commission on Wednesday launched the agency's first high-band 5G spectrum auction as it works to clear space for next-generation faster networks.

# Thank You

Haifeng Xu

University of Virginia

hx4ad@virginia.edu