

# Announcements: Homework I Out

- HW1 and a latex template for solutions are out on the course website: <http://www.haifeng-xu.com/cs6501fa19>
  - The HW sol template is for your convenience, but **not required**. Feel free to use your own template
- Due in two weeks: Thursday **09/19 3:30 pm**, rightly before class
- Homework submission
  1. Submit your PDF to UVA-Collab (collab course website just up)
  2. And hand a hard-copy over to Jing or Minbiao before class
- Start it early, and hope you enjoy it!

# CS650 I: Topics in Learning and Game Theory (Fall 2019)

## Introduction to Game Theory (I)

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Instructor: Haifeng Xu

# Outline

- Games and its Basic Representation
- Nash Equilibrium and its Computation
- Other (More General) Classes of Games

# (Recall) Example 1: Prisoner's Dilemma

- Two members A,B of a criminal gang are arrested
- They are questioned in two separate rooms
  - ❖ No communications between them

		B	
		B stays silent	B betrays
A	A stays silent	-1, -1	-3, 0
	A betrays	0, -3	-2, -2

Q: How should each prisoner act?

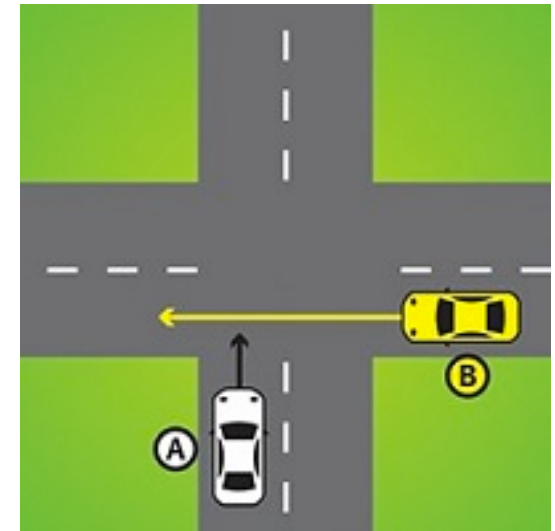
- Both of them betray, though (-1,-1) is better for both



## Example 2: Traffic Light Game

- Two cars heading to orthogonal directions

		B	
		STOP	GO
A	STOP	$(-3, -2)$	$(-3, 0)$
	GO	$(0, -2)$	$(-100, -100)$



Q: what are the equilibrium statuses?

Answer: (STOP, GO) and (GO, STOP)

# Example 3: Rock-Paper-Scissor

		Player 2		
		Rock	Paper	Scissor
Player 1	Rock	(0, 0)	(-1, 1)	(1, -1)
	Paper	(1, -1)	(0, 0)	(-1, 1)
	Scissor	(-1, 1)	(1, -1)	(0, 0)

Q: what is an equilibrium?

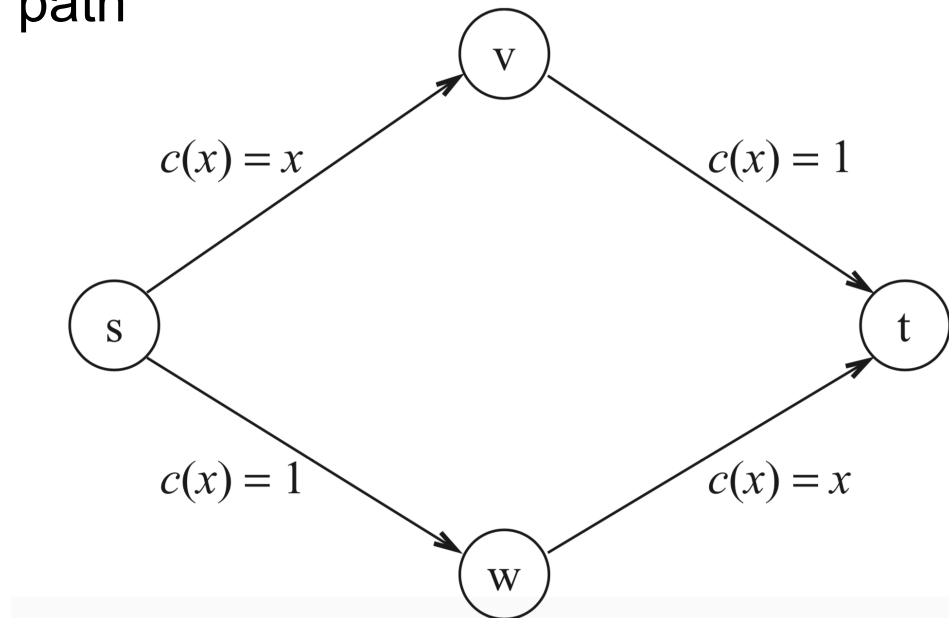
- Need to randomize – any deterministic action pair cannot make both players happy
- Common sense suggests  $(1/3, 1/3, 1/3)$

# Example 4: Selfish Routing

- One unit flow from  $s$  to  $t$  which consists of (infinite) individuals, each controlling an infinitesimal small amount of flow
- Each individual wants to minimize his own travel time

Q: What is the equilibrium status?

- Half unit flow through each path
- **Social cost** =  $3/2$

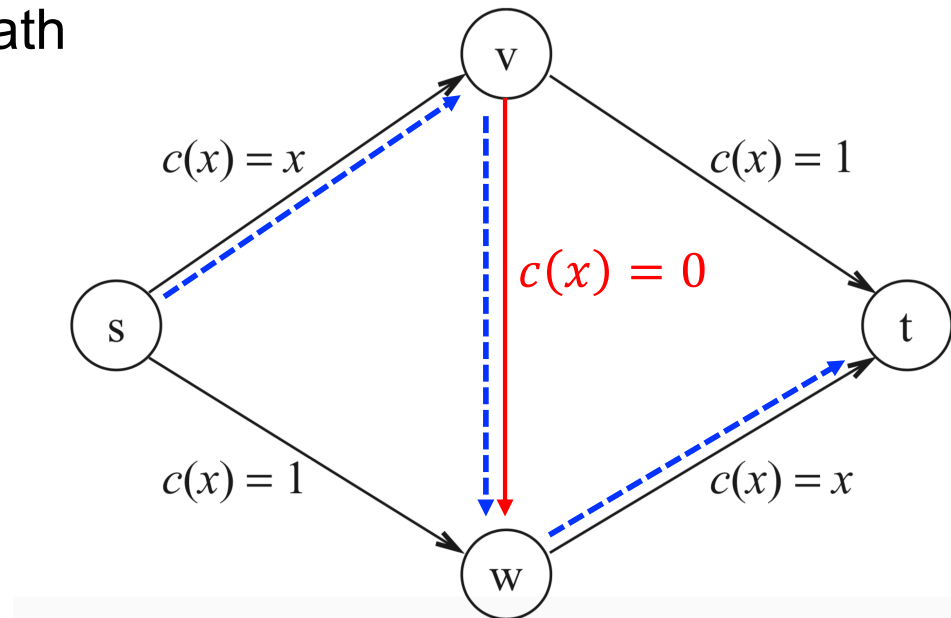


## Example 4: Selfish Routing

- One unit flow from  $s$  to  $t$  which consists of (infinite) individuals, each controlling an infinitesimal small amount of flow
- Each individual wants to minimize his own travel time

Q: What is the equilibrium status after adding a superior high way with 0 traveling cost?

- Everyone takes the blue path
- Social cost = 2



# Key Characteristics of These Games

- Each agent wants to maximize her own payoff
- An agent's payoff depends on other agents' actions
- The interaction stabilizes at a state where no agent can increase his payoff via **unilateral deviation**

# Strategic Games Are Ubiquitous

## ➤ Pricing

☐ Spirit Airlines (2) \$438

Departure time - Boston

☐ Morning (5:00am - 11:59am)

☐ Afternoon (12:00pm - 5:59pm)

☐ Evening (6:00pm - 11:59pm)

Arrival time - Chicago

☐ Early Morning (12:00am - 4:59am)

☐ Morning (5:00am - 11:59am)

☐ Afternoon (12:00pm - 5:59pm)

☐ Evening (6:00pm - 11:59pm)

6:30am - 8:15am

United

Very Good Flight (8.1/10)

[Details & baggage fees](#)

9:23am - 11:27am

American Airlines

Very Good Flight (8.3/10)

[Details & baggage fees](#)

7:01am - 9:10am

American Airlines

Very Good Flight (8.3/10)

[Details & baggage fees](#)

5:30am - 8:50am

Delta

Satisfactory Flight (6.4/10)

[Details & baggage fees](#)

2h 45m (Nonstop)

BOS - ORD

5 left at **\$236**

roundtrip

Select

3h 4m (Nonstop)

BOS - ORD

\$236

roundtrip

Select

3h 9m (Nonstop)

BOS - ORD

\$236

roundtrip

Select

4h 20m (1 stop)

BOS - 42m in DTW - ORD

1 left at **\$246**

roundtrip

Select

Run this search to

# Strategic Games Are Ubiquitous

- Pricing
- Sponsored search
  - Drives 90%+ of Google's revenue

The screenshot shows a Google search for "where to buy cruise vacation". The search results are dominated by sponsored ads. A red box highlights the first ad from Carnival Cruise Line, which includes a price of \$1.03. Below it, an ad from Expedia Cruises is shown with a price of \$1.02. Further down, an ad from VacationsToGo.com is shown with a price of \$0.60, and an ad from KAYAK is shown with a price of \$0.21. To the right of the main search results, there is a sponsored section titled "See cruise vac..." which features a 3-D Cruise Ship Centerpiece for \$0.65 (reduced from \$6.65) and a Zoom Party. The page also shows the Google logo, search bar, and navigation links like "All", "Shopping", "Images", "News", "Videos", "More", "Settings", and "Tools".

Google where to buy cruise vacation

About 103,000,000 results (0.63 seconds)

**Cruises | Caribbean Vacations | Carnival Cruise Line**  
**\$1.03**  
**Ad** [www.carnival.com/](http://www.carnival.com/) ▼  
Make Your Vacation Dreams A Reality With A Carnival® Cruise. Book Online Today! Signature Dining.  
**2-5 Day Cruises**  
Set Sail On These Quick Getaways  
That Fit Any Calendar, Anytime.  
**6-9 Day Cruises**  
Full-Length Cruises Mean More Time  
For Sun-Soaked Relaxation And Fun.

**Expedia Cruises | Cruise Vacations**  
**\$1.02**  
**Ad** [www.expedia.com/Cruises](http://www.expedia.com/Cruises) ▼  
Find the Perfect Cruise at the Best Price on Expedia, the #1 Travel Website. Last Minute Cruise Deals. Best Price Guaranteed. 4,000 Cruises Worldwide. Luxury Cruises Available. Destinations: Caribbean, Bahamas, Alaska, Mexico, Europe, Bermuda, Hawaii, Canada/New England.

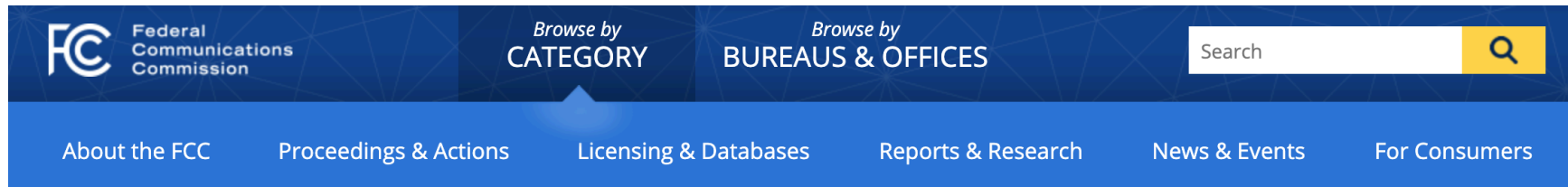
**2019 Cruises 82% Off | Compare All Cruise Lines | VacationsToGo.com**  
**\$0.60**  
**Ad** [www.vacationstogo.com/](http://www.vacationstogo.com/) ▼  
Book today for best price and selection on 2019 cruises. Save up to 82% Off Every Ship.  
Last-Minute Cruise Deals · Age 55+ Discounts · Caribbean up to 82% Off · Huge Carnival Deals

**KAYAK® Cruise Search | Find the Cheapest Cruise Deals | kayak.com**  
**\$0.21**  
**Ad** [www.kayak.com/vacations-go/last](http://www.kayak.com/vacations-go/last) ▼  
Rating for kayak.com: 3.8 - 605 reviews

See cruise vac... Sponsored ⓘ  
3-D Cruise Ship Centerpiece  
\$6.65 **\$0.65**  
Zoom Party  
→ More on Google

# Strategic Games Are Ubiquitous

- Pricing
- Sponsored search
  - Drives 90%+ of Google's revenue
- FCC's Allocation of spectrum to radio frequency users



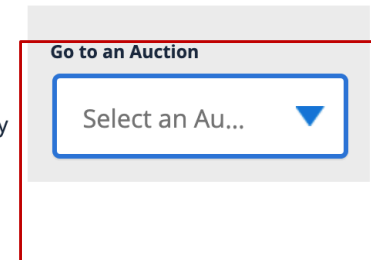
Home / Economics and Analytics /

## Auctions

### Proceedings & Actions

[Proceedings and Actions Overview](#)

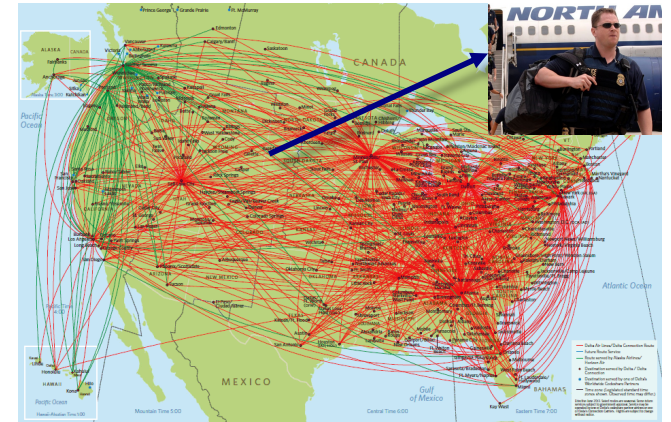
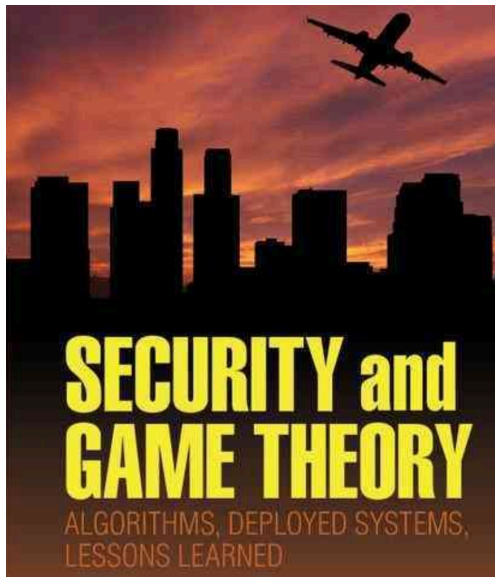
Since 1994, the Federal Communications Commission (FCC) has conducted auctions of licenses for electromagnetic spectrum. These auctions are open to any eligible company or individual that submits an application and upfront payment, and is found to be a qualified bidder by the Commission ([More About Auctions...](#))





# Strategic Games Are Ubiquitous

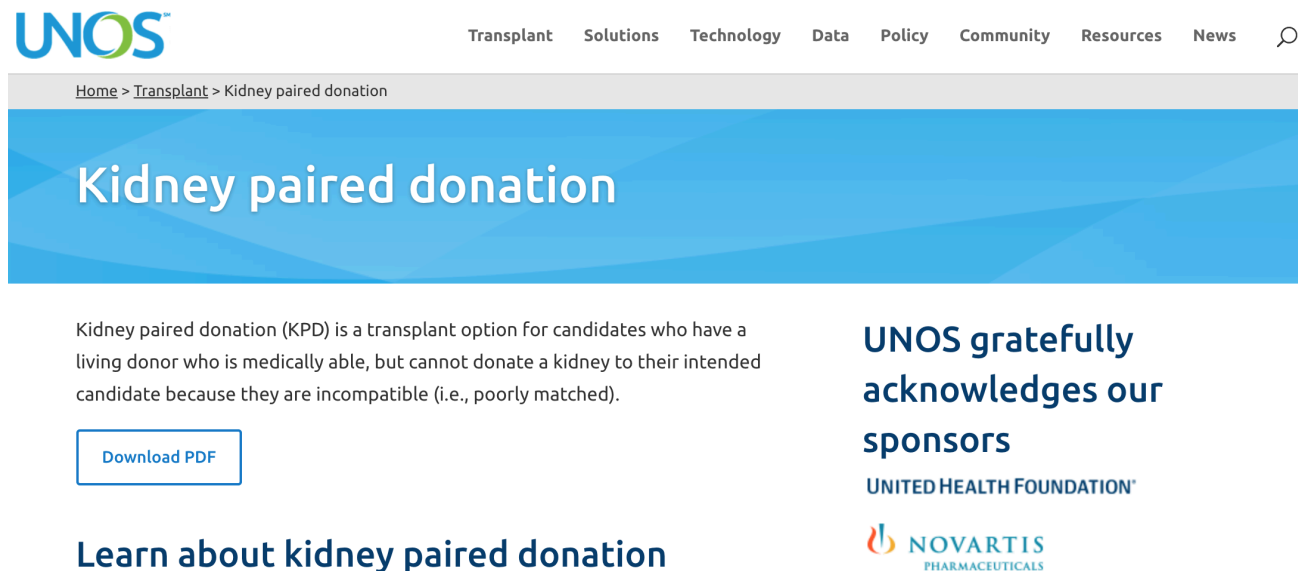
- Pricing
- Sponsored search
  - Drives 90%+ of Google's revenue
- FCC's Allocation of spectrum to radio frequency users
- National security, boarder patrolling, counter-terrorism



Optimize resource allocation against  
attackers/adversaries

# Strategic Games Are Ubiquitous

- Pricing
- Sponsored search
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- FCC's Allocation of spectrum to radio frequency users
- National security, boarder patrolling, counter-terrorism
- Kidney exchange – decides who gets which kidney at when



The screenshot shows the UNOS website's page for "Kidney paired donation". The header includes the UNOS logo and a navigation menu with links: Transplant, Solutions, Technology, Data, Policy, Community, Resources, and News. Below the header is a breadcrumb trail: Home > Transplant > Kidney paired donation. The main heading "Kidney paired donation" is displayed in a large, white font on a blue background. Below this, a paragraph explains that Kidney paired donation (KPD) is a transplant option for candidates who have a living donor who is medically able, but cannot donate a kidney to their intended candidate because they are incompatible (i.e., poorly matched). A "Download PDF" button is located below the paragraph. To the right, a section titled "UNOS gratefully acknowledges our sponsors" lists the "UNITED HEALTH FOUNDATION" and "NOVARTIS PHARMACEUTICALS". At the bottom left, a link says "Learn about kidney paired donation".

UNOS

Transplant Solutions Technology Data Policy Community Resources News

Home > Transplant > Kidney paired donation

## Kidney paired donation

Kidney paired donation (KPD) is a transplant option for candidates who have a living donor who is medically able, but cannot donate a kidney to their intended candidate because they are incompatible (i.e., poorly matched).

[Download PDF](#)

[Learn about kidney paired donation](#)

UNOS gratefully acknowledges our sponsors

UNITED HEALTH FOUNDATION\*

NOVARTIS  
PHARMACEUTICALS

# Strategic Games Are Ubiquitous

- Pricing
- Sponsored search
  - Drives 90%+ of Google's revenue
- FCC's Allocation of spectrum
- National security, border patrol
- Kidney exchange – decide who gets which
- Entertainment games: poker, blackjack, Go, chess . . .
- Social choice problems such as voting, fair division, etc.



# Strategic Games Are Ubiquitous

- Pricing
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- Entertainment games: poker, blackjack, Go, chess . . .
- Social choice problems such as voting, fair division, etc.

These are just a few example domains *where computer science has made significant impacts*; There are many others.

# Main Components of a Game

- **Players**: participants of the game, each may be an individual, organization, a machine or an algorithm, etc.
- **Strategies**: actions available to each player
- **Outcome**: the profile of player strategies
- **Payoffs**: a function mapping an outcome to a utility for each player

# Normal-Form Representation

- $n$  players, denoted by set  $[n] = \{1, \dots, n\}$
- Player  $i$  takes action  $a_i \in A_i$
- An outcome is the **action profile**  $a = (a_1, \dots, a_n)$ 
  - As a convention,  $a_{-i} = (a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n)$  denotes all actions excluding  $a_i$
- Player  $i$  receives payoff  $u_i(a)$  for any outcome  $a \in \prod_{i=1}^n A_i$ 
  - $u_i(a) = u_i(a_i, a_{-i})$  depends on other players' actions
- $\{A_i, u_i\}_{i \in [n]}$  are public knowledge

This is the most basic game model

- There are game models with richer and more intricate structures

# Illustration: Prisoner's Dilemma

- 2 players: 1 and 2
- $A_i = \{\text{silent}, \text{betray}\}$  for  $i = 1, 2$
- An outcome can be, e.g.,  $a = (\text{silent}, \text{silent})$
- $u_1(a), u_2(a)$  are pre-defined, e.g.,  $u_1(\text{silent}, \text{silent}) = -1$
- The whole game is public knowledge; players take actions simultaneously
  - Equivalently, take actions without knowing the others' actions

# Dominant Strategy

An action  $a_i$  is a **dominant strategy** for player  $i$  if  $a_i$  is better than any other action  $a'_i \in A_i$ , regardless what actions other players take.

Formally,

$$u_i(a_i, a_{-i}) \geq u_i(a'_i, a_{-i}), \quad \forall a'_i \neq a_i \text{ and } \forall a_{-i}$$

Note: “strategy” is just another term for “action”

A \ B	B stays silent	B betrays
A stays silent	-1, -1	-3, 0
A betrays	0, -3	-2, -2

Prisoner's Dilemma

- *Betray* is a dominant strategy for both
- Dominant strategies do not always exist
  - For example, the traffic light game

	STOP	GO
STOP	(-3, -2)	(-3, 0)
GO	(0, -2)	(-100, -100)



# Equilibrium

- An outcome  $a^*$  is an equilibrium if no player has incentive to deviate **unilaterally**. More formally,

$$u_i(a_i^*, a_{-i}^*) \geq u_i(a_i, a_{-i}^*), \quad \forall a_i \in A_i$$

- A special case of Nash Equilibrium, a.k.a., **pure strategy NE**
- If each player has a dominant strategy, they form an equilibrium
- But, an equilibrium does not need to consist of dominant strategies

		B	
		STOP	GO
A	STOP	(-3, -2)	(-3, 0)
	GO	(0, -2)	(-100, -100)

Traffic Light Game

# Equilibrium

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- A special case of Nash Equilibrium, a.k.a., *pure strategy NE*
- If each player has a dominant strategy, they form an equilibrium
- But, an equilibrium does not need to consist of dominant strategies

Pure strategy NE does not always exist...

	Rock	Paper	Scissor
Rock	(0, 0)	(-1, 1)	(1, -1)
Paper	(1, -1)	(0, 0)	(-1, 1)
Scissor	(-1, 1)	(1, -1)	(0, 0)

# Outline

- Games and its Basic Representation
- Nash Equilibrium and its Computation
- Other (More General) Classes of Games

# Pure vs Mixed Strategy

- Pure strategy: take an action deterministically
- Mixed strategy: can randomize over actions
  - Described by a distribution  $x_i$  where  $x_i(a_i) = \text{prob. of taking action } a_i$
  - $|A_i|$ -dimensional simplex  $\Delta_{A_i} := \{x_i : \sum_{a_i \in A_i} x_i(a_i) = 1, x_i(a_i) \geq 0\}$  contains all possible mixed strategies for player  $i$
  - Players draw their own actions *independently*
- Given strategy profile  $x = (x_1, \dots, x_n)$ , expected utility of  $i$  is

$$\sum_{a \in A} u_i(a) \cdot \prod_{i \in [n]} x_i(a_i)$$

- Often denoted as  $u(x)$  or  $u(x_i, x_{-i})$  or  $u(x_1, \dots, x_n)$
- When  $x_i$  corresponds to some pure strategy  $a_i$ , we also write  $u(a_i, x_{-i})$
- Fix  $x_{-i}$ ,  $u(x_i, x_{-i})$  is **linear** in  $x_i$

# Best Responses

Fix any  $x_{-i}$ ,  $x_i^*$  is called a best response to  $x_{-i}$  if

$$u_i(x_i^*, x_{-i}) \geq u_i(x_i, x_{-i}), \quad \forall x_i \in \Delta_{A_i}.$$

Claim. There always exists a pure best response

Proof: linear program “max  $u_i(x_i, x_{-i})$  subject to  $x_i \in \Delta_{A_i}$ ” has a vertex optimal solution

Remark: If  $x_i^*$  is a best response to  $x_{-i}$ , then any  $a_i$  in the support of  $x_i^*$  (i.e.,  $x_i^*(a_i) > 0$ ) must be equally good and are all pure best responses

# Nash Equilibrium (NE)

A mixed strategy profile  $x^* = (x_1^*, \dots, x_n^*)$  is a **Nash equilibrium** if

$$u_i(x_i^*, x_{-i}^*) \geq u_i(x_i, x_{-i}^*), \quad \forall x_i \in \Delta_{A_i}, \forall i \in [n].$$

That is, for any  $i$ ,  $x_i^*$  is a best response to  $x_{-i}^*$ .

## Remarks

- An equivalent condition:  $u_i(x_i^*, x_{-i}^*) \geq u_i(a_i, x_{-i}^*), \forall a_i \in A_i, \forall i \in [n]$ 
  - Since there always exists a pure best response
- It is not clear yet that such a mixed strategy profile would exist
  - Recall that pure strategy Nash equilibrium may not exist

# Nash Equilibrium (NE)

**Theorem (Nash, 1951):** Every finite game (i.e., finite players and actions) admits at least one mixed strategy Nash equilibrium.

- A foundational result in game-theory
- Example: rock-paper-scissor – what is a mixed strategy NE?
  - $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$  is a best response to  $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$

		1/3	1/3	1/3
		Rock	Paper	Scissor
ExpU = 0	Rock	(0, 0)	(-1, 1)	(1, -1)
ExpU = 0	Paper	(1, -1)	(0, 0)	(-1, 1)
ExpU = 0	Scissor	(-1, 1)	(1, -1)	(0, 0)

# Nash Equilibrium (NE)

**Theorem (Nash, 1951):** Every finite game (i.e., finite players and actions) admits at least one mixed strategy Nash equilibrium.

- An equilibrium outcome is not necessarily the best for players
  - Equilibrium only describes where the game stabilizes at
  - Many researches on understanding how self-interested behaviors reduces overall social welfare (recall the selfish routing game)
- A game may have many, even infinitely many, NEs
  - The issue of **equilibrium selection**

A \ B	B	
	B stays silent	B betrays
A stays silent	-1 / -1	0 / -3
A betrays	0 / -3	-2 / -2



# Intractability of Finding a NE

**Theorem:** Computing a Nash equilibrium for any two-player normal-form game is PPAD-hard.

Note: widely believed that PPAD-hard problems cannot be solved in poly time

- A two player game can be described by  $2mn$  numbers –  $u_1(i, j)$  and  $u_2(i, j)$  where  $i \in [m]$  is player 1's action and  $j \in [n]$  is player 2's.
- Theorem implies no  $\text{poly}(mn)$  time algorithm to compute an NE for any input game
- Ok, so what can we hope?
  - If the game has good structures, maybe we can find an NE efficiently
  - For example, zero-sum ( $u_1(i, j) + u_2(i, j) = 0$  for all  $i, j$ ), some resource allocation games

# An Exponential-Time Alg for Two-Player Nash

- What if we know the support of the NE:  $S_1, S_2$  for player 1 and 2?
- The NE can be formulated by a **linear feasibility** problem with variables  $x_1^*, x_2^*, U_1, U_2$

$$\forall j \in S_2: \quad \sum_{i \in S_1} u_2(i, j) x_1^*(i) = U_2$$

$$\forall j \notin S_2: \quad \sum_{i \in S_1} u_2(i, j) x_1^*(i) \leq U_2$$

$$\sum_{i \in [m]} x_1^*(i) = 1$$

$$\forall i \notin S_1: \quad x_1^*(i) = 0$$

$$\forall i \in [m]: \quad x_1^*(i) \geq 0$$

Symmetric constraints for player 2

- The challenge of computing a NE is to find the correct supports
  - No general tricks, typically just try all possibilities
  - Some pre-processing may help, e.g., eliminating dominated actions
- This approach does not work for  $> 2$  players games (why?)

# Intractability of Finding “Best” NE

**Theorem:** It is NP-hard to compute the NE that maximizes the sum of players' utilities or any single player's utility even in two-player games.

- Proofs of these results for NEs are beyond the scope of this course

# Outline

- Games and its Basic Representation
- Nash Equilibrium and its Computation
- Other (More General) Classes of Games

# Bayesian Games

- Previously, assumed players have complete knowledge of the game
- What if players are uncertain about the game?
- Can be modeled as a Bayesian belief about the state of the game
  - This is typical in Bayesian decision making, but not the only way

A \ B	B	
	B stays silent	B betrays
A stays silent	$\theta - 1$ $-1 + \theta$	$0$ $-3 + \theta$
A betrays	$0$ $\theta - 3$	$-2$ $-2$

*I will give an additional reward  $\theta$  for whoever staying silent*



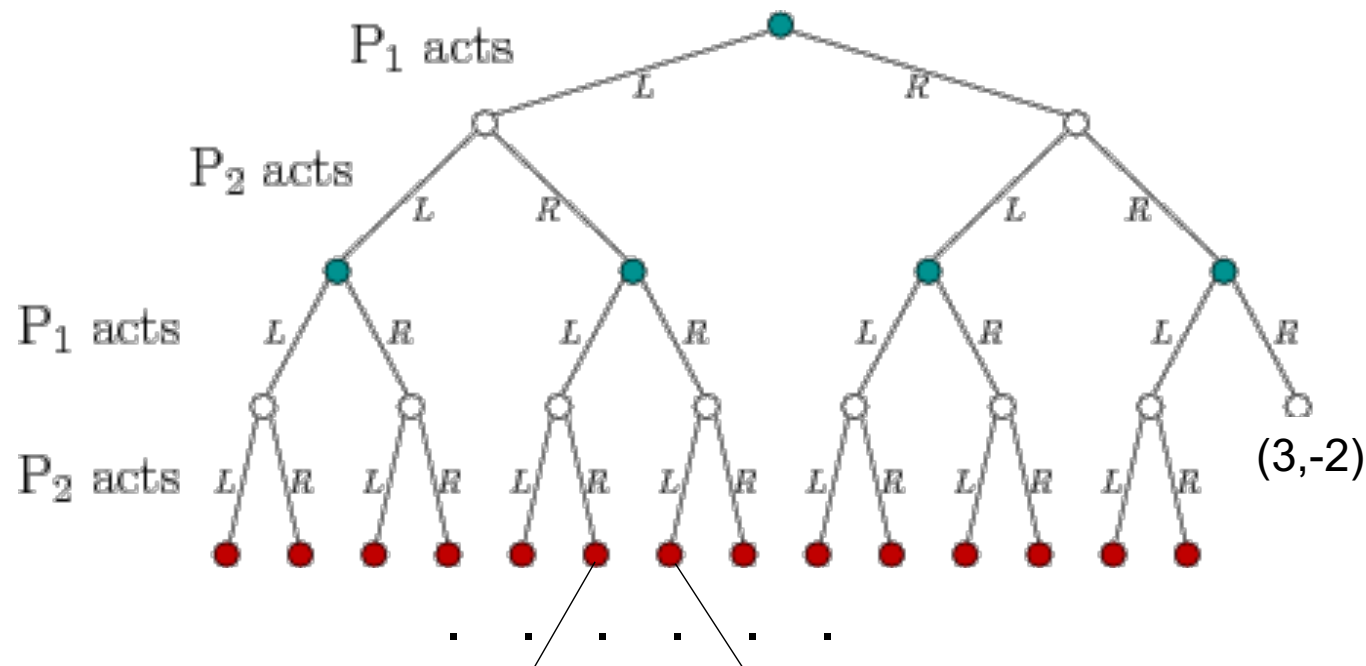
- It is believed that  $\theta \in \{0, 2, 4\}$  uniformly at random
- Or maybe the two players have different beliefs about  $\theta$

# Bayesian Games

- Previously, assumed players have complete knowledge of the game
- What if players are uncertain about the game?
- Can be modeled as a Bayesian belief about the state of the game
  - This is typical in Bayesian decision making, but not the only way
- More generally, can model player  $i$ ' payoffs as  $u_i^\theta$  where  $\theta$  is a **random** state of the game
- Each player obtains a (random) signal  $s_i$  that is correlated with  $\theta$ 
  - A joint prior distribution over  $(\theta, s_1, \dots, s_n)$  is assumed the public knowledge
- Can define a similar notion as Nash equilibrium, but expected utility also incorporates the randomness of the state of the game  $\theta$
- Applications: poker, blackjack, auction design, etc.

# Extensive-Form Games (EFGs)

- Previously, assumed players move only once and **simultaneously**
- More generally, can move sequentially and for multiple rounds
- Modeled by extensive-form game, described by a **game tree**



# Extensive-Form Games (EFGs)

- Previously, assumed players move only once and **simultaneously**
- More generally, can move sequentially and for multiple rounds
- Modeled by extensive-form game, described by a **game tree**
- EFGs are extremely general, can represent almost all kinds of games, but of course very difficult to solve



# A Remark

Sequential move fundamentally differs from simultaneous move

Nash equilibrium is only for simultaneous move

# A Remark

Sequential move fundamentally differs from simultaneous move

Nash equilibrium is only for simultaneous move

- What is an NE?
  - $(a_2, b_2)$  is the unique Nash, resulting in utility pair (1,2)
- If A moves first; B sees A's move and then best responds, how should A play?
  - Play action  $a_1$  deterministically!

		B	
		$b_1$	$b_2$
A	$a_1$	(2, 1)	(-2, -2)
	$a_2$	(2.01, -2)	(1, 2)

This sequential game model is called **Stackelberg game**, originally used to model market competition and now adversarial attacks.

# Thank You

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