CS6501: Topics in Learning and Game Theory
(Fall 2019)

Intro to Online Learning

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Outline

- Online Learning/Optimization
- Measure Algorithm Performance via Regret
- Warm-up: A Simple Example

Overview of Machine Learning

- Supervised learning

Labeled training data → ML Algorithm → Classifier/ Regression function

- Unsupervised learning

Unlabeled training data → ML Algorithm → Clusters/ Knowledge

- Semi-supervised learning (a combination of the two)

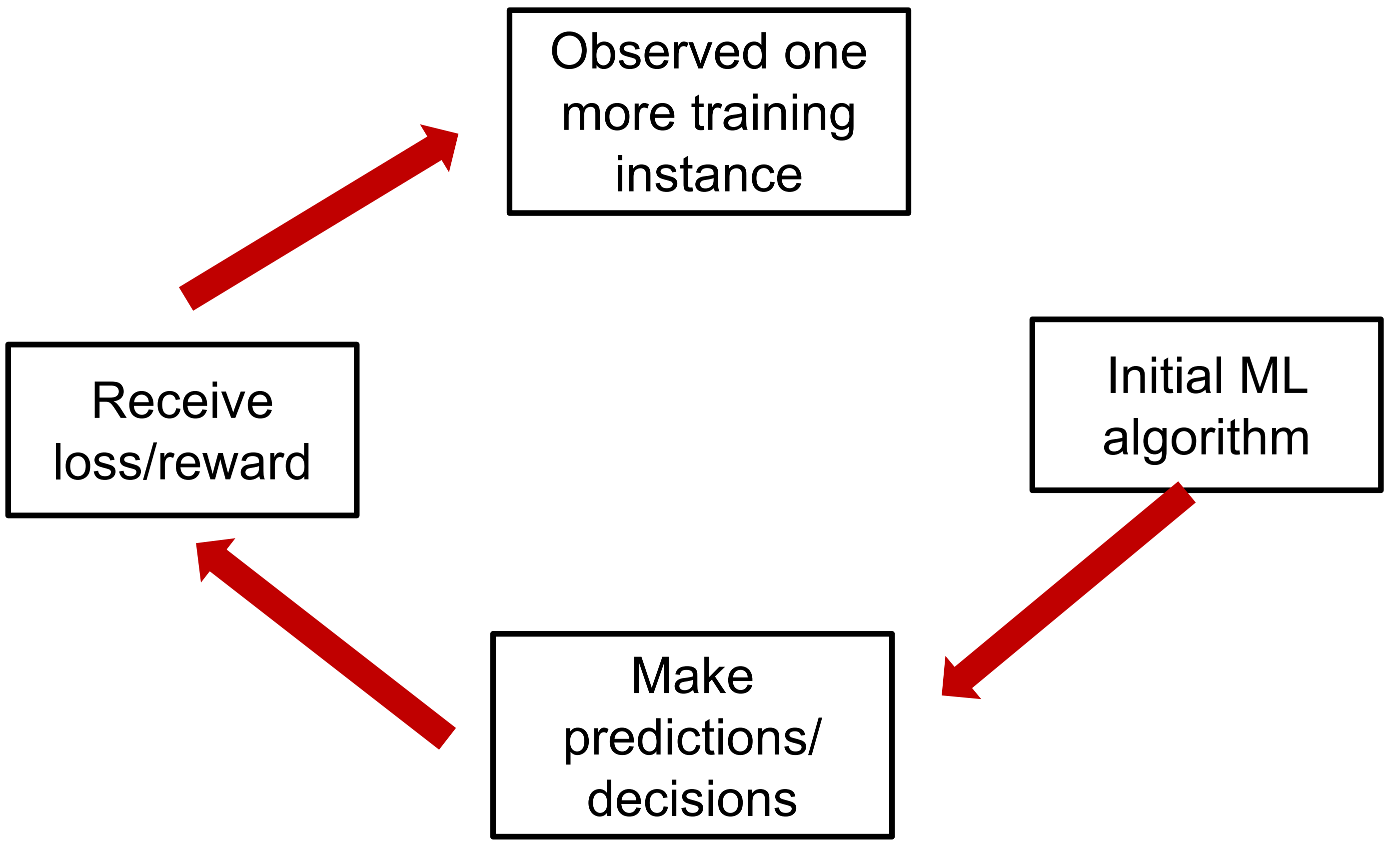
What else are there?

Overview of Machine Learning

- Supervised learning
- Unsupervised learning
- Semi-supervised learning
- Online learning
- Reinforcement learning
- Active learning
- . . .

Online Learning: When Data Come Online

The online learning pipeline



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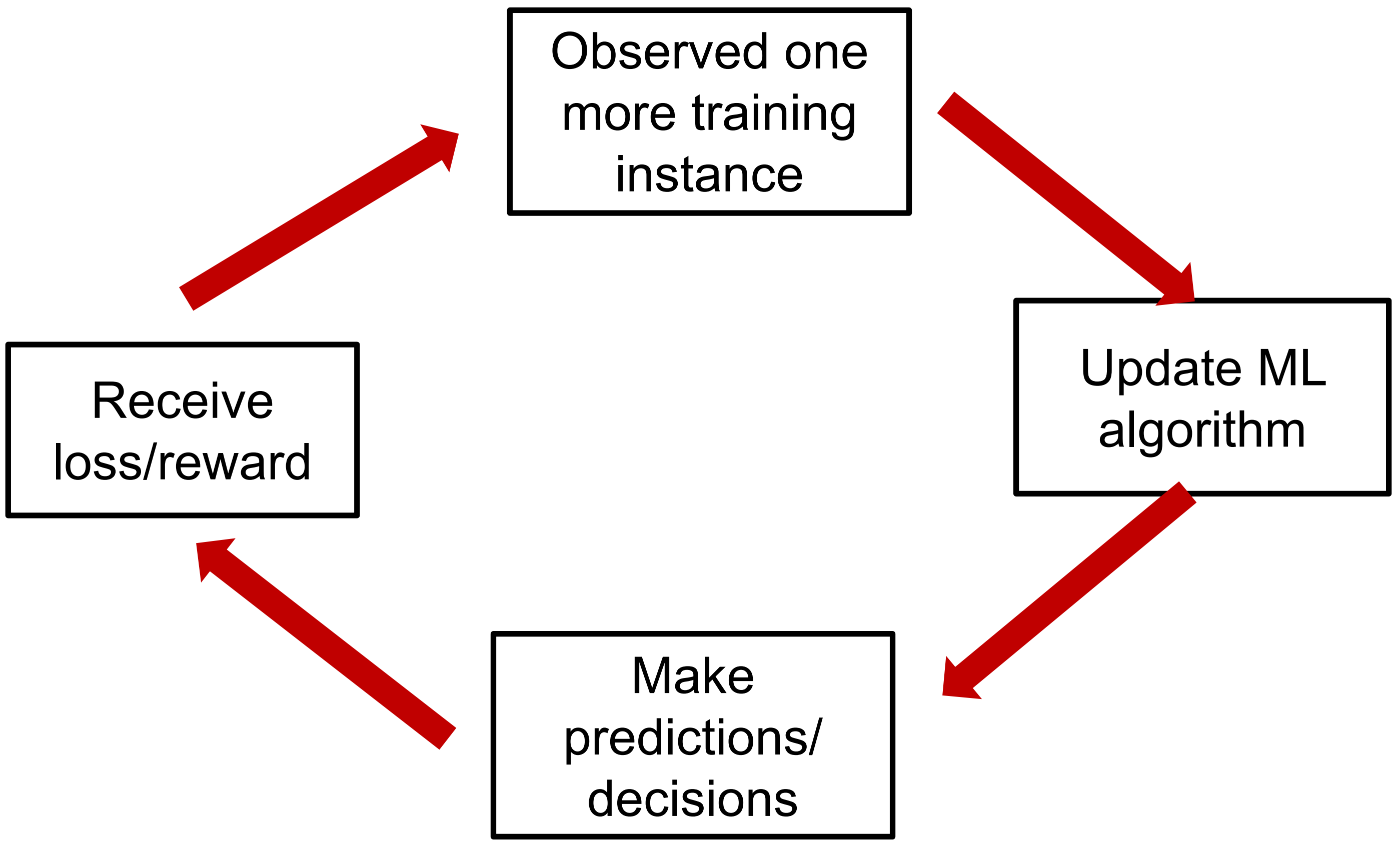


Typical Assumptions on Data

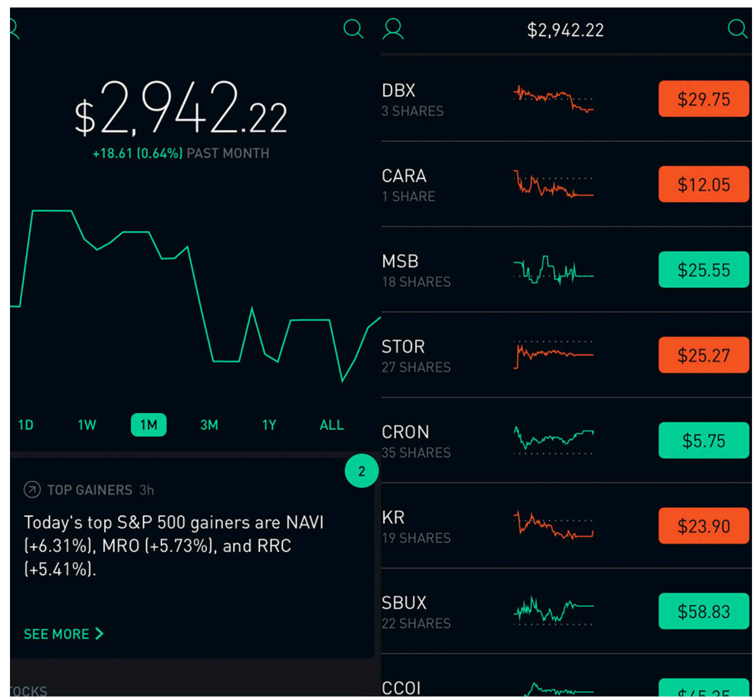
- Statistical feedback: instances drawn from a fixed distribution
  - Image classification, predict stock prices, choose restaurants, gambling machine (a.k.a., bandits)

- Adversarial feedback: instances are drawn adversarially
  - Spam detection, anomaly detection, game playing

- Markovian feedback: instances drawn from a distribution which is dynamically changing
  - Interventions, treatments

Online learning for Decision Making

- ➢ Learn to commute to school
  - Bus, walking, or driving? Which route? Uncertainty on the way?
- ➢ Learn to gamble or buy stocks



Online learning for Decision Making

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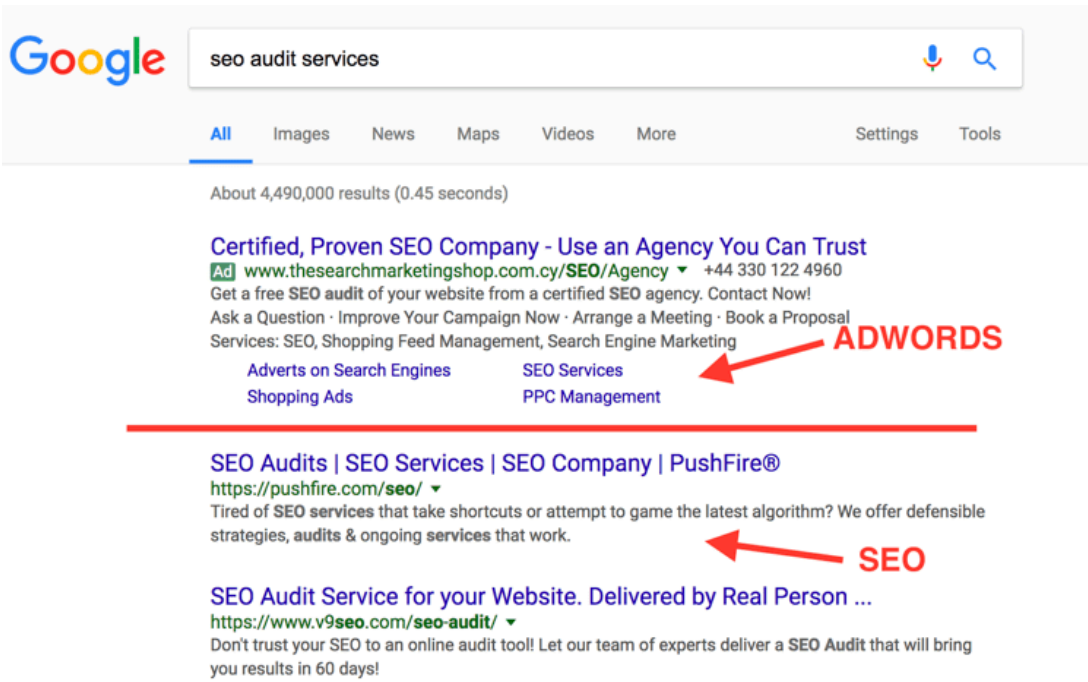
- Learn to gamble or buy stocks
- Advertisers learn to bid for keywords



Online learning for Decision Making

- Learn to commute to school
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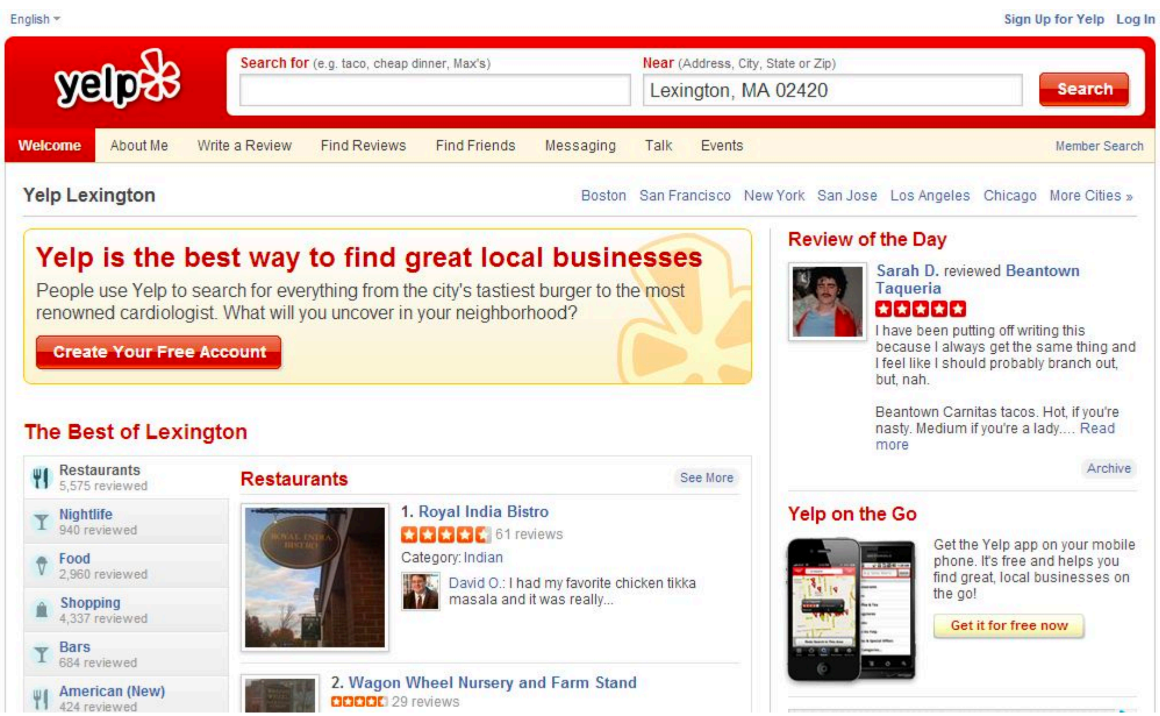

Online learning for Decision Making

- Learn to commute to school
  - Bus, walking, or driving? Which route? Uncertainty on the way?
- Learn to gamble or buy stocks
- Advertisers learn to bid for keywords
- Recommendation systems learn to make recommendations
- Clinical trials
- Robotics learn to react
- Learn to play games (video games and strategic games)
- Even how you learn to make decisions in your life
- . . .

Model Sketch

- A learner acts in an uncertain world for T time steps
- Each step $t = 1, \cdots, T$, learner takes action $i_t \in [n] = \{1, \cdots, n\}$
- Learner observes cost vector c_t where $c_t(i) \in [0,1]$ is the cost of action $i \in [n]$
  - Learner suffers cost $c_t(i_t)$ at step t
  - Can be similarly defined as reward instead of cost, not much difference
  - There are also "partial feedback" models (will not cover here)
- Adversarial feedbacks: c_t is chosen by an adversary
  - The powerful adversary has access to all the history (learner actions, past costs, etc.) until $t - 1$ and also the learner's algorithm
  - There are models of stochastic feedbacks (will not cover here)
- Learner's goal: minimize $\sum_{t \in [T]} c_t(i_t)$

Formal Procedure of the Model

At each time step $t = 1, \cdots, T$, the following occurs in order:

1. Learner picks a distribution p_t over actions $[n]$
2. Adversary picks cost vector $c_t \in [0,1]^n$ (he knows p_t)
3. Action $i_t \sim p_t$ is chosen and learner incurs cost $c_t(i_t)$
4. Learner observes c_t (for use in future time steps)

- Learner tries to pick distribution sequence $p_1, \cdots, p_T$ to minimize expected cost $\mathbb{E}\left[\sum_{t \in T} c_t(i_t)\right]$
  - Expectation over randomness of action
- The adversary does not have to really exist – it is assumed mainly for the purpose of worst-case analysis

Well, Adversary Seems Too Powerful?

- Adversary can choose $c_t \equiv 1, \forall t$; learner suffers cost T regardless
  - Cannot do anything non-trivial? We are done?
- If $c_t \equiv 1\ \forall t$, if you look back at the end, you do not regret anything – had you known such costs in hindsight, you cannot do better
  - From this perspective, cost T in this case is not bad

So what is a good measure for the performance of an online learning algorithm?

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Regret

- Measures how much the learner regrets, had he known the cost vector $c_1, \cdots, c_T$ in hindsight
- Formally,

$$R_T = \mathbb{E}_{i_t \sim p_t} \sum_{t \in [T]} c_t\,(i_t) - \boxed{\min_{i \in [n]} \sum_{t \in [T]} c_t(i)}$$

- Benchmark $\min_{i \in [n]} \sum_t c_t(i)$ is the learner utility had he known $c_1, \cdots, c_T$ and is allowed to take the best single action across all rounds

Regret

- Measures how much the learner regrets, had he known the cost vector $c_1, \cdots, c_T$ in hindsight
- Formally,

$$R_T = \mathbb{E}_{i_t \sim p_t} \sum_{t \in [T]} c_t(i_t) - \min_{i \in [n]} \sum_{t \in [T]} c_t(i)$$

- Benchmark $\min_{i \in [n]} \sum_t c_t(i)$ is the learner utility had he known $c_1, \cdots, c_T$ and is allowed to take the best single action across all rounds
  - There are other concepts of regret, e.g., swap regret (coming later)
  - But, $\min_{i \in [n]} \sum_t c_t(i)$ is mostly used

Regret is an appropriate performance measure of online algorithms

- It measures exactly the loss due to not knowing the data in advance

Average Regret

$$\bar{R}_T = \frac{R_T}{T} = \mathbb{E}_{i_t \sim p_t} \frac{1}{T} \sum_{t \in [T]} c_t\,(i_t) - \min_{i \in [n]} \frac{1}{T} \sum_{t \in [T]} c_t(i)$$

- When $\bar{R}_T \to 0$ as $T \to \infty$, we say the algorithm has vanishing regret or no-regret; the algorithm is called a no-regret online learning algorithm
  - Equivalently, R_T is sublinear in T
  - Both are used, depending on your habits

Our goal: design no-regret algorithms by minimizing regret

A Naive Strategy: Follow the Leader (FTL)

- That is, pick the action with the smallest accumulated cost so far

What is the worst-case regret of FTL?

Answer: worst (largest) regret $T/2$

- Consider following instance with 2 actions

t	1	2	3	4	5	. . .	T
$c_t(1)$	1	0	1	0	1	$\dots$	$*$
$c_t(2)$	0	1	0	1	0	$\dots$	$*$

- FTL always pick the action with cost 1 → total cost T
- Best action in hindsight has cost at most $T/2$

Randomization is Necessary

In fact, any deterministic algorithm suffers (linear) regret $(\mathrm{n}-1)T/n$

- Recall, adversary knows history and learner's algorithm
  - So he can infer our p_t at time t (but do **not** know our sampled $i_t \sim p_t$)
- But if p_t is deterministic, action i_t can also be inferred
- Adversary simply sets $c_t(i_t) = 1$ and $c_t(i) = 0$ for all $i \neq i_t$
- Learner suffers total cost T
- Best action in hindsight has cost at most T/n

Can randomized algorithm achieve sublinear regret?

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Consider a Simpler (Special) Setting

- Only two types of costs, $c_t(i) \in \{0,1\}$
- One of the actions is perfect – it always has cost 0
  - Minimum cost in hindsight is thus 0
  - Learner does not know which action is perfect

Is it possible to achieve sublinear regret in this simpler setting?

A Natural Algorithm

Observations:

1. If an action ever had non-zero costs, it is not perfect
2. Actions with all zero costs so far, we do not really know how to distinguish them currently

These motivate to the following natural algorithm

For $t = 1, \cdots, T$

- Identify the set of actions with zero total cost so far, and pick one action from the set uniformly at random.

Note: there is always at least one action to pick since the perfect action is always a candidate

Analysis of the Algorithm

- Fix a round t, we examine the expected loss from this round
- Let S_{good} = {actions with zero total cost before t} and $k = |S_{good}|$
  - So each action in S_{good} is picked with probability $1/k$

Analysis of the Algorithm

- Fix a round t, we examine the expected loss from this round
- Let S_{good} = {actions with zero total cost before t} and $k = |S|$
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- For any parameter $\epsilon \in [0,1]$, one of the following two happens
  - **<u>Case 1:</u>**
  - **<u>Case 2</u>**:

Analysis of the Algorithm

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- For any parameter $\epsilon \in [0,1]$, one of the following two happens
  - **Case 1:** at most ϵk actions from S_{good} have cost 1, in which case we suffer expected cost at most ϵ
  - **Case 2**:

Analysis of the Algorithm

- Fix a round t, we examine the expected loss from this round
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- For any parameter $\epsilon \in [0,1]$, one of the following two happens
  - **Case 1:** at most ϵk actions from S_{good} have cost 1, in which case we suffer expected cost at most ϵ
  - **Case 2**: at least ϵk actions from S_{good} have cost 1, in which case we suffer expected cost at most 1

Analysis of the Algorithm

- Fix a round t, we examine the expected loss from this round
- Let S_{good} = {actions with zero total cost before t} and $k = |S|$
  - So each action in S_{good} is picked with probability $1/k$
- For any parameter $\epsilon \in [0,1]$, one of the following two happens
  - **<u>Case 1:</u>** at most ϵk actions from S_{good} have cost 1, in which case we suffer expected cost at most ϵ
  - **<u>Case 2</u>**: at least ϵk actions from S_{good} have cost 1, in which case we suffer expected cost at most 1
- How many times can **Case 2** happen?
  - Each time it happens, size of S_{good} shrinks from k to at most $(1 - \epsilon)k$
  - At most $\log_{1-\epsilon} n^{-1}$ times
- The total cost of the algorithm is at most $\mathrm{T} \times \epsilon + \log_{1-\epsilon} n^{-1} \times 1$

Analysis of the Algorithm

- The cost upper bound can be further bounded as follows

$$
\begin{aligned}
\text{Total Cost} &\leq \; T\times\epsilon + \log_{1-\epsilon} n^{-1} \times 1 \\
&= \; T\epsilon + \frac{\ln n}{-\ln(1-\epsilon)} && \text{Since } \log_a b = \frac{\ln b}{\ln a} \\
&\leq \; T\epsilon + \frac{\ln n}{\epsilon} && \text{Since } -\ln(1-\epsilon) \geq \epsilon, \forall \epsilon \in (0,1)
\end{aligned}
$$

- The above upper bound holds for any ϵ, so picking $\epsilon = \sqrt{\ln n / T}$ we have

$$R_T = \text{Total Cost} \leq \; 2\sqrt{T \ln n}$$

Sublinear in T 

What about the General Case?

- $c_t \in [0,1]^n$
- No perfect action
- Previous algorithm can be re-written in a more "mathematically beautiful" way, which turns out to generalize

For $t = 1, \cdots, T$
- Identify the set of actions with zero total cost so far, and pick one action from the set uniformly at random.

What about the General Case?

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Initialize weight $w_1(i) = 1, \forall i = 1, \cdots n$
For $t = 1, \cdots, T$
1. Let $W_t = \sum_{i \in [n]} w_t(i)$, pick action i with probability $w_t(i)/W_t$
2. Observe cost vector $c_t \in \{0,1\}^n$
3. Update $w_{t+1}(i) = w_t(i) \cdot (1 - c_t(i))$

What about the General Case?

- $c_t \in [0,1]^n$ → the weight update process is still okay
- No perfect action
- Previous algorithm can be re-written in a more "mathematically beautiful" way, which turns out to generalize

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What about the General Case?

- $c_t \in [0,1]^n$ → the weight update process is still okay
- No perfect action → more conservative when eliminating actions
- Previous algorithm can be re-written in a more "mathematically beautiful" way, which turns out to generalize

Initialize weight $w_1(i) = 1, \forall i = 1, \cdots n$
For $t = 1, \cdots, T$
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3. Update $w_{t+1}(i) = w_t(i) \cdot (1 - \epsilon \cdot c_t(i))$

Multiplicative Weight Update (MWU)

Theorem. Multiplicative Weight Update (MWU) achieves regret at most $O(\sqrt{T \ln n})$ for the previously described general setting.

- Proof of the theorem is left to the next lecture
- Note: we really care about theoretical bound for online algorithms
  - The environment is uncertain and difficult to simulate, there is no easy way to experimentally evaluate the algorithm

Is $O(\sqrt{T \ln n})$ is best possible regret?

Next, we show $\sqrt{T \ln n}$ is tight

Lower Bound I

$(\ln n)$ term is necessary

- Consider any $T \approx \ln(n-1)$
- Will construct a series of random costs such that there is a perfect action yet any algorithm will have expected cost $T/2$
  - At $t = 1$, randomly pick half actions to have cost 1 and remaining actions have cost 0
  - At $t = 2, 3, \cdots, T$: among perfect actions so far, randomly pick half of them to have cost 1 and remaining actions have cost 0
- Since $T < \ln(n)$, at least one action remains perfect at the end
- But any algorithm suffers expected cost $1/2$ at each round (why?); The total cost will be $T/2$
- Costs are stochastic, not adversarial? → Will be provably worse when costs become adversarial
  - Just FYI: A formal proof is by Yao's minimax principle

Lower Bound 2

$(\sqrt{T})$ term is necessary

- Consider 2 actions only, still stochastic costs
- For $t = 1, \cdots, T$, cost vector $c_t = (0,1)$ or $(1,0)$ uniformly at random
  - c_t's are independent across t's
- Any algorithm has 50% chance of getting cost 1 at each round, and thus suffers total expected cost $T/2$
- What about the best action in hindsight?
  - From action 1's perspective, its costs form a $0-1$ bit sequence, each bit drawn independently and uniformly at random
  - $c[1] = \sum_{t \in T} c_t(1)$ is $Binomial(T, \frac{1}{2})$ and $c(2) = T - c[1]$
  - The cost of best action in hindsight is $\min(c[1], T - c[1])$
  - $\mathbb{E} \min(c[1], T - c[1]) = \frac{T}{2} - \Theta(\sqrt{T})$

Thank You

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