Announcements

- HW 1 is due now

CS6501: Topics in Learning and Game Theory
(Fall 2019)

Adversarial Multi-Armed Bandits

Instructor: Haifeng Xu

Outline

- The Adversarial Multi-armed Bandit Problem
- A Basic Algorithm: Exp3
- Regret Analysis of Exp3

Recap: Online Learning So Far

Setup: T rounds; the following occurs at round t:

1. Learner picks a distribution p_t over actions $[n]$
2. Adversary picks cost vector $c_t \in [0,1]^n$
3. Action $i_t \sim p_t$ is chosen and learner incurs cost $c_t(i_t)$
4. Learner observes c_t (for use in future time steps)

Performance is typically measured by regret:

$$R_T = \sum_{i\in[n]} \sum_{t\in[T]} c_t(i)\, p_t(i) - \min_{j\in[n]} \sum_{t\in[T]} c_t(j)$$

The multiplicative weight update algorithm has regret $O(\sqrt{T \ln n})$.

Recap: Online Learning So Far

Convergence to equilibrium

- In repeated zero-sum games, if both players use a no-regret learning algorithm, their average strategy converges to an NE
- In general games, the average strategy converges to a CCE

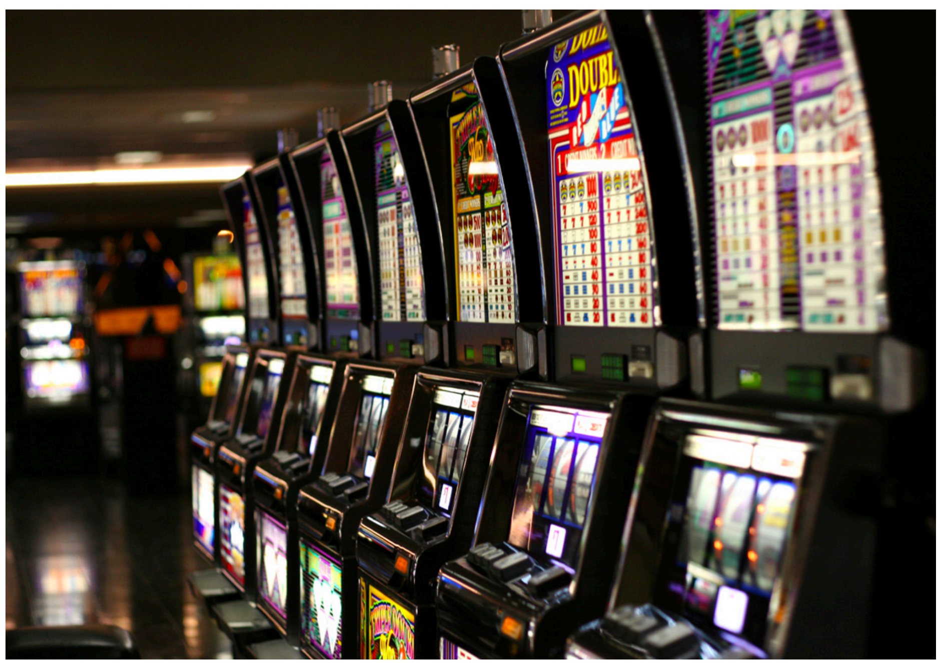
Swap regret – a “stronger” regret concept and better convergence

- Def: each action i has a chance to deviate to another action $s(i)$
- In repeated general games, if both players use a no-swap-regret learning algorithm, their average strategy converges to a CE

There is a general reduction, converting any learning algorithm with regret R to one with swap regret nR.

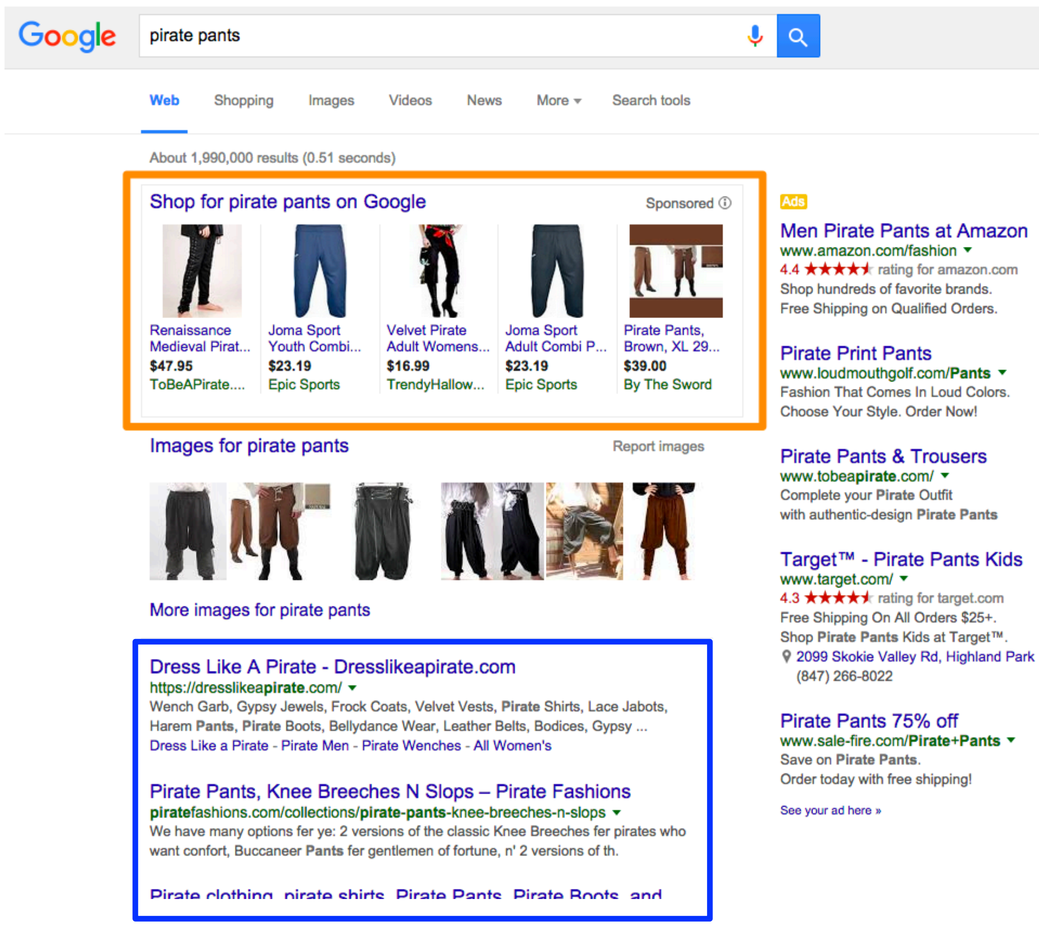
This Lecture: Address Partial Feedback

- In online learning, the whole cost vector c_t can be observed by the learner, despite she only takes a single action i_t
  - Realistic in some applications, e.g., stock investment
- In many cases, we only see the reward of the action we take
  - For example: slot machines, a.k.a., multi-armed bandits



Other Applications with Partial Feedback

- Online advertisement placement or web ranking
  - Action: ad placement or ranking of webs
  - Cannot see the feedback for untaken actions



Other Applications with Partial Feedback

- Online advertisement placement or web ranking
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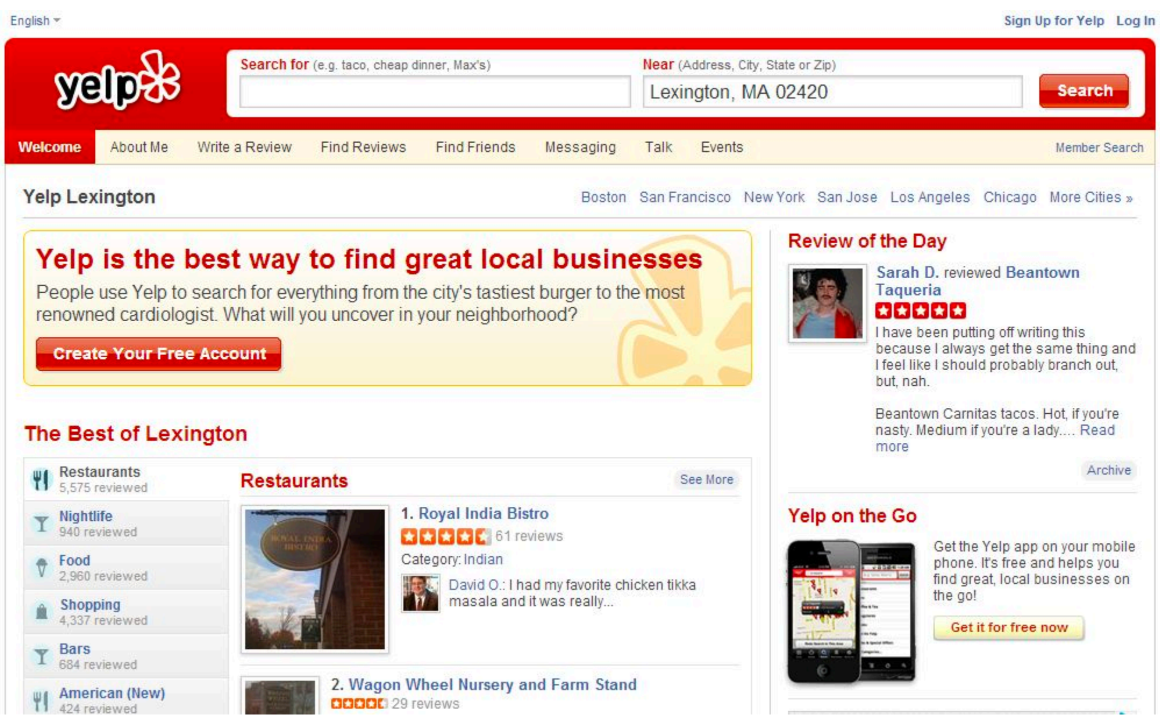
- Recommendation system:
  - Action = recommended option (e.g., a restaurant)
  - Do not know other options’ feedback



Other Applications with Partial Feedback

- Online advertisement placement or web ranking
  - Action: ad placement or ranking of webs
  - Cannot see the feedback for untaken actions
- Recommendation system:
  - Action = recommended option (e.g., a restaurant)
  - Do not know other options' feedback
- Clinical trials
  - Action = a treatment
  - Don't know what would happen for treatments not chosen
- Playing strategic games
  - Cannot observe opponents' strategies but only know the payoff of the taken action
  - E.g., Poker games, competition in markets

Adversarial Multi-Armed Bandits (MAB)

- Very much like online learning, except partial feedback
  - The name "bandit" is inspired by slot machines
- Model: at each time step $t = 1, \cdots, T$; the following occurs in order
  1. Learner picks a distribution p_t over arms $[n]$
  2. Adversary picks cost vector $c_t \in [0,1]^n$
  3. Arm $i_t \sim p_t$ is chosen and learner incurs cost $c_t(i_t)$
  4. Learner only observes $c_t(i_t)$ (for use in future time steps)
- Though we cannot observe c_t, adversary still picks c_t before i_t is sampled

Q: since learner does not observe $c_t(i)$ for $i \neq i_t$, can adversary arbitrarily modify these $c_t(i)$'s after i_t has been selected?

No, because this makes c_t depends on sampled i_t which is not allowed

Outline

- The Adversarial Multi-armed Bandit Problem
- A Basic Algorithm: Exp3
- Regret Analysis of Exp3

Recall the algorithm for full information setting:

Parameter: ϵ

Initialize weight $w_1(i) = 1, \forall i = 1, \cdots n$

For $t = 1, \cdots, T$

1. Let $W_t = \sum_{i \in [n]} w_t(i)$, pick arm i with probability $w_t(i)/W_t$
2. Observe cost vector $c_t \in [0,1]^n$
3. For all $i \in [n]$, update $w_{t+1}(i) = w_t(i) \cdot (1 - \epsilon c_t(i))$

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3. For all $i \in [n]$, update $w_{t+1}(i) = w_t(i) \cdot e^{-\epsilon \cdot c_t(i)}$

Recall $1 - \delta \approx e^{-\delta}$ for small δ

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- In this lecture we will use this exponential-weight variant, and prove its regret bound en route
- Also called *Exponential Weight Update (EWU)*

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Basic idea of Exp3

- Want to use EWU, but do not know vector c_t → try to estimate c_t!
- Well, we really only have $c_t(i_t)$, what can we do?

Estimate $\bar{c_t} = (0, \cdots, 0, c_t(i_t), 0, \cdots 0)^T$? ✗ Too optimistic

Estimate $\bar{c_t} = \left(0, \cdots, 0, \frac{c_t(i_t)}{p_t(i_t)}, 0, \cdots 0\right)^T$ ✓

Exp3: a Basic Algorithm for Adversarial MAB

Parameter: ϵ

Initialize weight $w_1(i) = 1, \forall i = 1, \cdots n$

For $t = 1, \cdots, T$

1. Let $W_t = \sum_{i \in [n]} w_t(i)$, pick arm i with probability $w_t(i)/W_t$
2. Observe cost vector $c_t \in [0,1]^n$
3. For all $i \in [n]$, update $w_{t+1}(i) = w_t(i) \cdot e^{-\epsilon \cdot \bar{c}_t(i)}$ where $\bar{c}_t = (0, \cdots, 0, c_t(i_t)/p_t(i_t), 0, \cdots 0)^T$.

- That is, weight is updated only for the pulled arm
  - Because we really don't know how good are other arms at t
  - But i_t is more heavily penalized now
  - Attention: $c_t(i_t)/p_t(i_t)$ may be extremely large if $p_t(i_t)$ is small
- Called Exp3: Exponential-weight algorithm for Exploration and Exploitation

A Closer Look at the Estimator $\overline{c_t}$

- $\overline{c_t}$ is random – it depends on the randomly sampled $i_t \sim p_t$
- $\overline{c_t}$ is an unbiased estimator of c_t, i.e., $\mathbb{E}_{i_t \sim p_t} \overline{c_t} = c_t$
  - Because given p_t, for any i we have

$$\mathbb{E}_{i_t \sim p_t} \overline{c_t}(i) = \mathbb{P}(i_t = i) \cdot \frac{c_t(i)}{p_t(i)} + \mathbb{P}(i_t \neq i) \cdot 0$$

$$= p_t(i) \cdot \frac{c_t(i)}{p_t(i)}$$

$$= c_t(i)$$

- This is exactly the reason for our choice of $\overline{c_t}$

Regret

$$R_T = \sum_{i\in[n]} \sum_{t\in[T]} c_t(i)\, p_t(i) - \min_{j\in[n]} \sum_{t\in[T]} c_t(j)$$

Some key differences from online learning

- R_T is random (even it already takes expectation over $i_t \sim p_t$)
  - Because distribution p_t itself is random, depends on sampled $i_1, \cdots i_{t-1}$
  - That is, if we run the same algorithm for multiple times, we will get different R_T value even when facing the same adversary!

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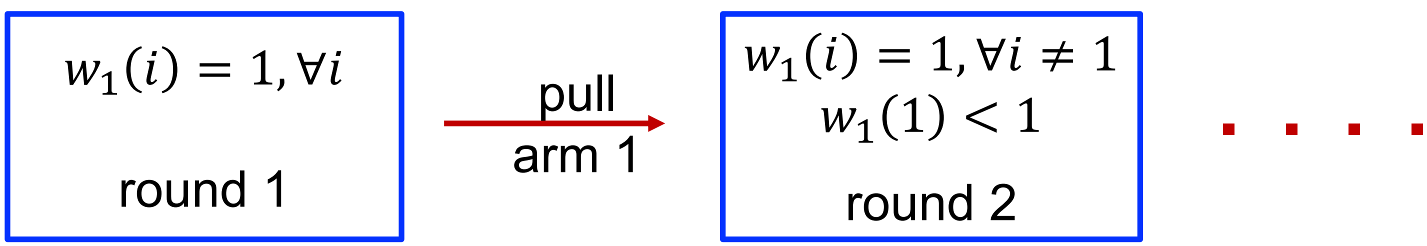

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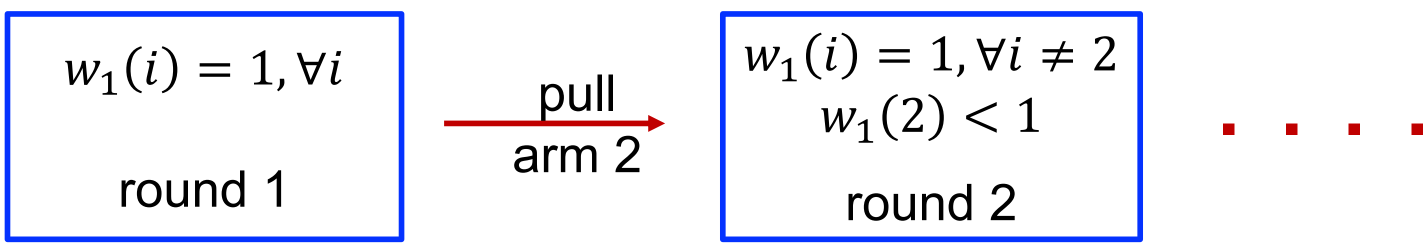

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Some key differences from online learning

- R_T is random (even it already takes expectation over $i_t \sim p_t$)
  - Because distribution p_t itself is random, depends on sampled $i_1, \cdots i_{t-1}$
  - That is, if we run the same algorithm for multiple times, we will get different R_T value even when facing the same adversary
- Cost vector c_t is also random as it generally depends on p_t
  - Adversary maps distribution p_t to a cost vector c_t
- This is not the case in online learning
  - If we run the same algorithm for multiple times, we shall obtain the same R_T value if facing the same adversary

Regret

$$R_T = \sum_{i \in [n]} \sum_{t \in [T]} c_t(i)\, p_t(i) - \min_{j \in [n]} \sum_{t \in [T]} c_t(j)$$

- Therefore, in principle, we have to upper bound $\mathbb{E}(R_T)$ where expectation is over the randomness of arm sampling

$$\mathbb{E}(R_T) = \mathbb{E}\left[\sum_{i \in [n]} \sum_{t \in [T]} c_t(i)\, p_t(i) - \min_{j \in [n]} \sum_{t \in [T]} c_t(j)\right]$$

$$= \sum_{i \in [n]} \sum_{t \in [T]} \mathbb{E}[c_t(i) p_t(i)] - \mathbb{E}\left[\min_{j \in [n]} \sum_{t \in [T]} c_t(j)\right]$$

by linearity of expectation

Regret

$$R_T = \sum_{i\in[n]} \sum_{t\in[T]} c_t(i)\, p_t(i) - \min_{j\in[n]} \sum_{t\in[T]} c_t(j)$$

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$$\geq \sum_{i\in[n]} \sum_{t\in[T]} \mathbb{E}[c_t(i)p_t(i)] - \min_{j\in[n]} \sum_{t\in[T]} \mathbb{E}[c_t(j)]$$

because $\min_{j\in[n]} \sum_{t\in[T]} \mathbb{E}[c_t(j)] \geq \mathbb{E}\left[\min_{j\in[n]} \sum_{t\in[T]} c_t(j)\right]$

(proof: homework exercise)

Regret

$$R_T = \sum_{i\in[n]} \sum_{t\in[T]} c_t(i)\, p_t(i) - \min_{j\in[n]} \sum_{t\in[T]} c_t(j)$$

- Therefore, in principle, we have to upper bound $\mathbb{E}(R_T)$ where expectation is over the randomness of arm sampling

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$$\geq \underbrace{\sum_{i\in[n]} \sum_{t\in[T]} \mathbb{E}[c_t(i) p_t(i)] - \min_{j\in[n]} \sum_{t\in[T]} \mathbb{E}[c_t(j)]}_{\text{Pseudo-Regret } \overline{R_T}}$$

- Good regret guarantees good pseudo-regret, but not the reverse

Bounding regret turns out to be challenging

- Exp3 is not sufficient to guarantee small regret
- Next, we instead prove that Exp3 has small pseudo-regret
  - As is typical in many works
- A slight modification of Exp3 can be proved to have small regret

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Theorem. The pseudo regret of Exp3 is $O(\sqrt{\mathrm{nT}\ln n})$.

High-level idea of the proof

- Pretend to be in the full information setting with cost equal the estimated $\overline{c_t}$
- Relate $\overline{c_t}$ to c_t since we know it is an unbiased estimator of c_t

Imitate a Full-Info Setting with Cost $\bar{c}_t$

- Recall regret bound for full information setting

$$R_T^{full} \leq \frac{\ln n}{\epsilon} + \epsilon T$$

- This holds for any cost vector, thus also $\bar{c}_t$
- But…one issue is that $\bar{c}_t(i_t)$ may be greater than 1
- Not a big issue – the same analysis yields the following bound

$$R_T^{full} \leq \frac{\ln n}{\epsilon} + \epsilon \max_i \sum_{t\in[T]} [\bar{c}_t(i)]^2$$

Real Issue: $\bar{c}_t(i)$ may be too large that we cannot bound R_T^{full}

Imitate a Full-Info Setting with Cost $\overline{c_t}$

A regret bound as follows turns out to work for our proof

$$R_T^{full} \leq \frac{\ln n}{\epsilon} + \epsilon \sum_t \sum_i p_t(i) \, [\overline{c_t}(i)]^2$$

- That is, instead of $\max_i$, the bound here averages over i
- Why more useful?
  - The $p_t(i)$ term will help to cancel out a $p_t(i)$ demominator in $\overline{c_t}(i) = c_t(i)/p_t(i)$
  - This turns out to be enough to bound the regret

Step 1: Tighter Regret for Full-Info Case

Lemma 1. The regret of the following algorithm is at most $\frac{\ln n}{\epsilon} + \frac{\epsilon}{2}\sum_t \sum_i p_t(i)\,[\overline{c_t}(i)]^2$ for any cost vector $\overline{c_t} \geq 0$.

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For $t = 1, \cdots, T$

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2. Observe cost vector $\overline{c_t} \geq 0$
3. For all $i \in [n]$, update $w_{t+1}(i) = w_t(i) \cdot e^{-\epsilon \cdot \overline{c_t}(i)}$

Note: this yields a bound $\frac{\ln n}{\epsilon} + \frac{\epsilon}{2} T$ when $c_t \in [0,1]^n$

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Proof: similar technique – carefully bound certain quantity

- Consider quantity $\sum_{i\in[n]} p_t(i) e^{-\epsilon c_t(i)}$

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Proof: similar technique – carefully bound certain quantity

- Consider quantity $\sum_{i\in[n]} p_t(i) e^{-\epsilon c_t(i)}$

Why this term?

- It tracks weight decrease (will be clear in next slide)
- The algebraic reasons, $e^{-\delta} \approx 1 - \delta + \delta^2/2$, which will give rise to both the term $p_t(i)\overline{c_t}(i)$ and $p_t(i)[\overline{c_t}(i)]^2$

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➢ Consider quantity $\sum_{i\in[n]} p_t(i) e^{-\epsilon c_t(i)}$

Fact 1. $\sum_{i\in[n]} p_t(i) e^{-\epsilon c_t(i)} = W_{t+1}/W_t$, where $W_t = \sum_i w_t(i)$.

- The term $\sum_{i\in[n]} p_t(i) e^{-\epsilon c_t(i)}$ is the decreasing rate of W_t
- Formal proof: HW exercise

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- Formal proof: HW exercise

Corollary. $\sum_t \log\left[\sum_{i\in[n]} p_t(i) e^{-\epsilon c_t(i)}\right] = \log W_{T+1} - \log n$

- Telescope sum and $W_1 = n$

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Fact 2. $\sum_t \log\left[\sum_{i\in[n]} p_t(i) e^{-\epsilon c_t(i)}\right] \leq -\epsilon \sum_{t,i} p_t(i) c_t(i) + \frac{\epsilon^2}{2} \sum_{t,i} p_t(i)[c_t(i)]^2$.

Follows from algebraic calculation

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Follows from algebraic calculation

$$\sum_t \log\left[\sum_{i\in[n]} p_t(i)e^{-\epsilon c_t(i)}\right] \leq \sum_t \log\left[\sum_{i\in[n]} p_t(i)\left[1 - \epsilon c_t(i) + \frac{\epsilon^2}{2}[c_t(i)]^2\right]\right]$$

By $e^{-\delta} \leq 1 - \delta + \delta^2/2$

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$$= \sum_t \log\left[1 - \sum_{i\in[n]} p_t(i)\epsilon c_t(i) + \sum_{i\in[n]} p_t(i)\frac{\epsilon^2}{2}[c_t(i)]^2\right]$$

Since $\sum_{i\in[n]} p_t(i) = 1$

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Follows from algebraic calculation

$$\sum_t \log\left[\sum_{i\in[n]} p_t(i) e^{-\epsilon c_t(i)}\right] \leq \sum_t \log\left[\sum_{i\in[n]} p_t(i) [1 - \epsilon c_t(i) + \frac{\epsilon^2}{2} [c_t(i)]^2]\right]$$

$$= \sum_t \log\left[1 - \sum_{i\in[n]} p_t(i) \epsilon c_t(i) + \sum_{i\in[n]} p_t(i) \frac{\epsilon^2}{2} [c_t(i)]^2\right]$$

$$\leq -\epsilon \sum_{t,i} p_t(i) c_t(i) + \frac{\epsilon^2}{2} \sum_{t,i} p_t(i) [c_t(i)]^2$$

Since $\log(1+\delta) \leq \delta$ for any δ

Step 1: Tighter Regret for Full-Info Case

Lemma 1. The regret of the following algorithm is at most $\frac{\ln n}{\epsilon} + \frac{\epsilon}{2} \sum_t \sum_i p_t(i) \, [\overline{c_t}(i)]^2$ for any cost vector $\overline{c_t} \geq 0$.

- Consider quantity $\sum_{i \in [n]} p_t(i) e^{-\epsilon c_t(i)}$
- Combining the two facts yields the lemma
  - HW exercise

Step 2: Relate $\overline{c_t}$ to Pseudo-Regret

Recall pseudo-regret definition

$$\overline{R_T} = \sum_{t\in[T]} \mathbb{E}[c_t \cdot p_t] - \min_{j\in[n]} \sum_{t\in[T]} \mathbb{E}[c_t(j)]$$

$$= \max_{j\in[n]} \left[\sum_{t\in[T]} \mathbb{E}[c_t \cdot p_t] - \sum_{t\in[T]} \mathbb{E}[c_t(j)] \right]$$

$$= \max_{j\in[n]} \underbrace{\sum_{t\in[T]} \mathbb{E}[c_t \cdot p_t - c_t(j)]}_{\text{Pseudo-regret from action } j}$$

Step 2: Relate $\overline{c_t}$ to Pseudo-Regret

Lemma 2. $\sum_{t\in[T]} \mathbb{E}[c_t \cdot p_t - c_t(j)] = \sum_{t\in[T]} \mathbb{E}[\overline{c_t} \cdot p_t - \overline{c_t}(j)]$

- That is, expected pseudo regret from j w.r.t. true cost c_t equals that w.r.t. the estimated cost $\overline{c_t}$

Recall pseudo-regret definition

$$\overline{R_T} = \sum_{t\in[T]} \mathbb{E}[c_t \cdot p_t] - \min_{j\in[n]} \sum_{t\in[T]} \mathbb{E}[c_t(j)]$$

$$= \max_{j\in[n]} \left[\sum_{t\in[T]} \mathbb{E}[c_t \cdot p_t] - \sum_{t\in[T]} \mathbb{E}[c_t(j)] \right]$$

$$= \max_{j\in[n]} \underbrace{\sum_{t\in[T]} \mathbb{E}[c_t \cdot p_t - c_t(j)]}_{\text{Pseudo-regret from action } j}$$

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- Proof:

$$\mathbb{E}[\overline{c_t} \cdot p_t \; - \overline{c_t}(j)] = \mathbb{E}\big[\mathbb{E}[\overline{c_t} \cdot p_t \; - \overline{c_t}(j) | \, p_t]\big]$$

Because the randomness of $\overline{c_t}$ comes:

1. Randomness of $i_t \sim p_t$
2. Randomness of p_t itself which depends on $i_1, \cdots, i_{t-1}$

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Lemma 2. $\sum_{t\in[T]} \mathbb{E}[c_t \cdot p_t - c_t(j)] = \sum_{t\in[T]} \mathbb{E}[\overline{c_t} \cdot p_t - \overline{c_t}(j)]$

- Proof:

$$\mathbb{E}[\overline{c_t} \cdot p_t - \overline{c_t}(j)] = \mathbb{E}\big[\mathbb{E}[\overline{c_t} \cdot p_t - \overline{c_t}(j) \mid p_t]\big]$$

$$= \mathbb{E}\big[\mathbb{E}[c_t \cdot p_t - c_t(j) \mid p_t]\big]$$

Because conditioning on p_t, $\overline{c_t}$ is an unbiased estimator of c_t

Step 2: Relate $\overline{c_t}$ to Pseudo-Regret

Lemma 2. $\sum_{t\in[T]} \mathbb{E}[c_t \cdot p_t - c_t(j)] = \sum_{t\in[T]} \mathbb{E}[\overline{c_t} \cdot p_t - \overline{c_t}(j)]$

- Proof:

$$\mathbb{E}[\overline{c_t} \cdot p_t - \overline{c_t}(j)] = \mathbb{E}\big[\mathbb{E}[\overline{c_t} \cdot p_t - \overline{c_t}(j) | p_t]\big]$$

$$= \mathbb{E}\big[\mathbb{E}[c_t \cdot p_t - c_t(j) | p_t]\big]$$

$$= \mathbb{E}[c_t \cdot p_t - c_t(j)]$$

Step 3: Derive Pseudo-Regret Bounds

Lemma 1. The regret of the following algorithm is at most $\frac{\ln n}{\epsilon} + \frac{\epsilon}{2}\sum_t \sum_i p_t(i)\,[\overline{c_t}(i)]^2$ for any cost vector $\overline{c_t} \geq 0$.

Lemma 2. $\sum_{t\in[T]} \mathbb{E}[c_t \cdot p_t - c_t(j)] = \sum_{t\in[T]} \mathbb{E}[\overline{c_t} \cdot p_t - \overline{c_t}(j)]$

- For any j, we have

$$\sum_{t\in[T]} \mathbb{E}[c_t \cdot p_t - c_t(j)] = \mathbb{E}\left[\sum_{t\in[T]}[\overline{c_t} \cdot p_t - \overline{c_t}(j)]\right]$$

$$\leq \mathbb{E}\left[\frac{\ln n}{\epsilon} + \frac{\epsilon}{2}\sum_t \sum_i p_t(i)\,[\overline{c_t}(i)]^2\right]$$

By Lemma 1

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$$\leq \mathbb{E}\left[\frac{\ln n}{\epsilon} + \frac{\epsilon}{2}\sum_t \sum_i p_t(i)\,[\overline{c_t}(i)]^2\right]$$

$$= \frac{\ln n}{\epsilon} + \frac{\epsilon}{2}\mathbb{E}\left[\ \mathbb{E}\left[\sum_t \sum_i p_t(i)\,[\overline{c_t}(i)]^2\,|p_t\right]\ \right]$$

By conditional expectation

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- For any j, we have

$$
\begin{aligned}
\sum_{t\in[T]} \mathbb{E}[c_t \cdot p_t \; - c_t(j)] &= \mathbb{E}\left[\sum_{t\in[T]}[\overline{c_t} \cdot p_t \; - \overline{c_t}(j)]\right] \\
&\leq \mathbb{E}\left[\frac{\ln n}{\epsilon} + \frac{\epsilon}{2}\sum_t \sum_i p_t(i)\,[\overline{c_t}(i)]^2\right] \\
&= \frac{\ln n}{\epsilon} + \frac{\epsilon}{2}\mathbb{E}\left[\;\mathbb{E}\left[\sum_t \sum_i p_t(i)\,[\overline{c_t}(i)]^2 \,|p_t\right]\right] \\
&= \frac{\ln n}{\epsilon} + \frac{\epsilon}{2}\mathbb{E}\left[\sum_t \sum_i p_t(i)\,\mathbb{E}\left[[\overline{c_t}(i)]^2 | p_t\right]\right]
\end{aligned}
$$

By linearity of expectation

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$$\sum_{t\in[T]} \mathbb{E}[c_t \cdot p_t \ - c_t(j)] = \mathbb{E}\left[\sum_{t\in[T]}[\overline{c_t} \cdot p_t \ - \overline{c_t}(j)]\right]$$

$$\leq \mathbb{E}\left[\frac{\ln n}{\epsilon} + \frac{\epsilon}{2}\sum_t \sum_i p_t(i)\,[\overline{c_t}(i)]^2\right]$$

$$= \frac{\ln n}{\epsilon} + \frac{\epsilon}{2}\mathbb{E}[\ \mathbb{E}[\sum_t \sum_i p_t(i)\,[\overline{c_t}(i)]^2\ |p_t]\ ]$$

$$= \frac{\ln n}{\epsilon} + \frac{\epsilon}{2}\mathbb{E}\left[\sum_t \sum_i p_t(i)\,\mathbb{E}\left[[\overline{c_t}(i)]^2 | p_t\right]\right]$$

Observer $\mathbb{E}\left[[\overline{c_t}(i)]^2 | p_t\right] = 0 \cdot [1 - p_t(i)] + \left[\frac{c_t(i)}{p_t(i)}\right]^2 \cdot p_t(i) = \frac{[c_t(i)]^2}{p_t(i)}$

Step 3: Derive Pseudo-Regret Bounds

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$$\leq \mathbb{E}\left[\frac{\ln n}{\epsilon} + \frac{\epsilon}{2}\sum_t \sum_i p_t(i)\,[\bar{c_t}(i)]^2\right]$$

$$= \frac{\ln n}{\epsilon} + \frac{\epsilon}{2}\mathbb{E}[\,\mathbb{E}[\sum_t \sum_i p_t(i)\,[\bar{c_t}(i)]^2\,|p_t]\,]$$

$$= \frac{\ln n}{\epsilon} + \frac{\epsilon}{2}\mathbb{E}\left[\sum_t \sum_i \boxed{p_t(i)}\,\mathbb{E}[[\bar{c_t}(i)]^2|p_t]\right]$$

Observer $\mathbb{E}\left[[\bar{c_t}(i)]^2|p_t\right] = 0 \cdot [1 - p_t(i)] + \left[\frac{c_t(i)}{p_t(i)}\right]^2 \cdot p_t(i) = \frac{[c_t(i)]^2}{\boxed{p_t(i)}}$

Step 3: Derive Pseudo-Regret Bounds

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$$\begin{aligned}
\sum_{t\in[T]} \mathbb{E}[c_t \cdot p_t - c_t(j)] &= \mathbb{E}\left[\sum_{t\in[T]}[\bar{c_t} \cdot p_t - \bar{c_t}(j)]\right] \\
&\leq \mathbb{E}\left[\frac{\ln n}{\epsilon} + \frac{\epsilon}{2}\sum_t \sum_i p_t(i)\,[\bar{c_t}(i)]^2\right] \\
&= \frac{\ln n}{\epsilon} + \frac{\epsilon}{2}\mathbb{E}\left[\,\mathbb{E}\left[\sum_t \sum_i p_t(i)\,[\bar{c_t}(i)]^2\,|p_t\right]\,\right] \\
&= \frac{\ln n}{\epsilon} + \frac{\epsilon}{2}\mathbb{E}\left[\sum_t \sum_i p_t(i)\,\mathbb{E}\left[[\bar{c_t}(i)]^2|p_t\right]\right] \\
&= \frac{\ln n}{\epsilon} + \frac{\epsilon}{2}\mathbb{E}\left[\sum_t \sum_i [c_t(i)]^2\right] \\
&\leq \frac{\ln n}{\epsilon} + \frac{\epsilon}{2} nT
\end{aligned}$$

Pick $\epsilon = \sqrt{\frac{2\ln n}{nT}}$ yields a regret bound of $O(\sqrt{\mathrm{nT}\ln n})$

Summary of the Proof

- A tighter regret bound for full information setting
- Treat the (realized) estimated $\overline{c_t}$ as the cost for full information
- Expected pseudo-regret w.r.t. to c_t equals expected pseudo-regret w.r.t. to $\overline{c_t}$
- Upper bound pseudo-regret by taking expectation over $\overline{c_t}$'s

The True Regret and Beyond

- Exp3 does not guarantee good true regret, still because $c_t(i)/p_t(i)$ may be too large
  - Pseudo-regret "smooths out" $p_t(i)$ by taking expectations first
- To obtain good true regret, need to modify Exp3 by adding some uniform exploration so that $p_t(i)$ is never too small
  - More intricate analysis, but will get the same regret bound $O(\sqrt{\mathrm{nT}\ln n})$
- In additional to adversarial feedback, a "nicer" setting is when the cost of each arm is drawn from a fixed but unknown distribution
  - Called stochastic multi-armed bandits
  - Naturally, Exp3 and regret bound $O(\sqrt{\mathrm{nT}\ln n})$ still applies
  - But a better algorithm called Upper-Confidence Bounds (UCB) yields much better regret bound $O(\sqrt{\mathrm{n}\ln T})$
  - Different analysis techniques

Thank You

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