The Strange Role of Information Asymmetry in Auctions — Does More Accurate Value Estimation Benefit A Bidder?

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Abstract

We study the second-price auction in which bidders have asymmetric information regarding the item's value. Each bidder's value for the item depends on a *private* component and a *public* component. While each bidder observes their own private component, they hold different and asymmetric information about the public component. We characterize the equilibrium of this auction game and study how the asymmetric bidder information affects their equilibrium bidding strategies. We also discover multiple surprisingly counter-intuitive equilibrium phenomena. For instance, a bidder may be better off if she is less informed regarding the public component. Conversely, a bidder may sometimes be worse off if she obtains more accurate estimation about the auctioned item. Our results suggest that efforts devoted by bidders to improve their value estimations, as widely seen in today's online advertising auctions, may not always be to their benefit.

1 Introduction

Thanks to convenient access to information and the significant advances of machine learning techniques, we can now make unprecedentedly accurate predictions about various uncertain quantities. Naturally, these predictions have also changed how our decisions are made. One prominent examples, among many others, is the significant effort devoted by both Internet companies and online advertisers to predicting an Internet user's interest in their ads. Such predicted level of interest, typically characterized by the *click through rate* (Richardson, Dominowska, and Ragno 2007), will then affect the advertiser's value, and thus the bid, for the user's impression. More generally, when we make many other purchases (e.g., a house, used car or an artwork), we tend to collect as much data as possible in order to reduce our uncertainty about the item's value estimation.

An important question, however, is *whether such more accurate estimations always benefit us*. At a first glance, this answer might seem obvious — how could knowing more ever be harmful? Certainly, if an agent's payoff, with possible *uncertainty*, only depends on its own decision and is not affected by any other agent's actions (i.e., there are *no externalities*), then more accurate estimations always improve the agent's decision (Blackwell 1953). However, the answer turns out to be highly non-trivial once an agent's payoff has externalities, and specifically, is affected by other self-interested agents' actions in a *strategic* setup. This raises the following basic and intriguing question:

> *In real-world strategic settings with uncertainty, does more accurate estimation always benefit the agent?*

In this paper, we investigate the above question in the fundamental sealed-bid *second price auction* (a.k.a., Vickrey auction) which is widely used in the online advertising industry (Edelman, Ostrovsky, and Schwarz 2007).[1] Specifically, we consider a second-price auction with $n+1$ bidders $0, 1, \cdots, n$. The valuation of the auctioned item for each bidder i is the sum of a private component V_i and a public component C that is common to all bidders. While V_i's and C are all drawn from publicly known distributions, bidders have asymmetric information about the public component C; that is, (only) bidder 0 can observe the realized C. As a motivating example, consider an ad auction among insurance companies who all bid for an Internet user searching the keyword of "insurance". The common component C here captures the total profit any insurance company could earn from this particular user, which depends on the user's attributes and typically does not differ much across insurance companies. The private component V_i describes the specific operational cost of insurance company i. Bidder 0 could be a performance-driven advertiser who delegates her bidding to professional marketing agencies which indeed have much more accurate information about Internet users.

This work aims at understanding how the special bidder 0's extra information may affect the *equilibrium* as well as agents' *utilities* by comparing two different situations: (1) the *asymmetric* world as we described above; and (2) the *symmetric* world in which bidder 0 also cannot observe C.

Summary of results. As a necessary step, we first characterize the equilibrium of the above game under information asymmetry. Specifically, when there are only two bidders, 0 and 1, we provide an explicit characterization of the equilibrium under mild assumptions. As an application of this characterization, we prove that there exists a critical value t

[1]More precisely, second-price auctions are used when there is only a single ad slot for sale, e.g., for display ads (Korula, Mirrokni, and Nazerzadeh 2015).

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for the uninformed bidder 1 such that he bids more conservatively than his bid in the symmetric world whenever his value $v_1 \leq t$, but bids more aggressively when $v_1 \geq t$. For $n > 1$, we derive differential equations that characterize the bidding equilibrium. As an application of this characterization, we show that uninformed bidders' bids decrease as n increases; that is, more competition leads to more conservative bidding in the asymmetric world, whereas bidders' equilibrium bid remain the same in the symmetric world.

We then leverage our equilibrium characterization to discover multiple quite surprising facts, which illustrate the intricate role of information asymmetry in auctions. We show that the informed bidder 0 may be worse off in the asymmetric world — i.e., bidder 0's information advantage may be harmful to her. Surprisingly, we also find that an uninformed bidder may be better off in the asymmetric world — i.e., ignorance of her value may increase a bidder's utility at equilibrium. Finally, we observe that the auctioneer's revenue may decrease as the uninformed bidders learn additional information about C. This gives rise to a simple counter example to the celebrated "linkage principle" (Milgrom and Weber 1982), which posits that the auctioneer's revenue always increases as bidders become more informed. We remark that such a counter example was discovered recently in (Syrgkanis, Kempe, and Tardos 2015) as well, but with a much more involved analysis than ours.

Related work. The study of information asymmetry in auctions has a long history, dating back to the seminal work of Wilson (1967). Most previous works have focused on *first* price auction, and like us, have assumed that one bidder is fully informed whereas all other bidders are not informed (Wilson 1967; Milgrom and Weber 1982; Engelbrecht-Wiggans, Milgrom, and Weber 1983). However, our focus in this work is a different yet similarly basic auction format, the second price auction.

The seminal work of Milgrom and Weber (1982) proved the linkage principle (as mentioned above) for commonly seen auction formats with symmetric bidders and positively correlated (a.k.a., *associated*) bidder signals. Hausch (1987) and another interesting recent work (Syrgkanis, Kempe, and Tardos 2015) analyze the equilibrium of both first and second price auctions but with non-symmetric bidders. Both works consider only two bidders and demonstrate the failure of the linkage principle in their models. Most of these previous works assume common value. However, our model can have many bidders and uses a more general bidder value model (which includes the common value model as a special case with $V_i \equiv 0$ for each bidder i). On the other hand, the information structure in our model is simpler by assuming one bidder fully observes C while all others observe no information. Therefore, our results are not quite comparable. Moreover, these previous works are all primarily focused on the auctioneer's revenue, whereas we mainly study bidders' bidding behavior and utility changes at equilibrium. To our knowledge, this perspective has not been investigated much before.

Our work is also related to recent literature on designing signaling schemes to influence bidders' bidding strategy (Emek et al. 2012; Badanidiyuru, Bhawalkar, and Xu 2018; Li and Das 2019; Bergemann et al. 2021). However, these works also focuses on the auctioneer's revenue. Our work can be viewed as understanding the equilibrium under one particular signaling scheme, i.e., with full information revealed to exactly one bidder.

2 Preliminaries

Basic Setup. We consider a single-item second-price auction with $n+1$ bidders, denoted by set $[n] = \{0, 1, ..., n\}$, and *correlated* bidder values. Bidder i has value $V_i + C$ for the item where V_i is the *private value* that depends only on i, and C is the *common value* that is the same for every bidder. For any $i \in [n]$, V_i is a random variable drawn from a continuous distribution with strictly positive probability density function (PDF) $f_i(v)$ supported on $[l_i, u_i]$ for some $0 \leq l_i < u_i$. Let $F_i(v)$ be the corresponding cumulative distribution function (CDF). The common value C is also a random variable drawn from a continuous distribution with strictly positive PDF $g(c)$ supported on $[l_c, u_c]$ for $0 \leq l_c < u_c$; $G(c)$ is the CDF. By convention, we use capital letters (e.g., V_i) to denote random variables and small letters (e.g., v_i) denote the realization. As a standard assumption, V_i's and C are assumed to all be independent.

In the second-price auction, each bidder $i \in [n]$ submits a bid b_i. The highest bid wins, receives the auctioned item, and pays the second highest bid. Let i^* be the winning bidder and b_{i^-} be the second highest bid. The payoff of i^* is $v_{i^*}+c-b_{i^-}$ and the payoffs of all other bidders are 0.

Information Structure. All distributions are common knowledge, i.e., known to all bidders. Each bidder i can observe the realization of his own private value v_i. However, bidder 0 is the only special bidder who additionally observes the *realized* common value of c. So the game exhibits information asymmetry — bidder 0 has more accurate estimation of c. As widely known in the literature (Milgrom and Weber 1982), under information asymmetry, truthful bidding ceases to be an equilibrium in second-price auctions. Nevertheless, the following lemma shows that truthful bidding remains a dominant strategy for bidder 0 (but not for other bidders; see examples later). Its proof follows the same argument as the truthfulness proof in a standard VCG mechanism, and thus is omitted.

Lemma 1. *Truthful bidding is a dominant strategy for bidder* 0.

Consequently, we shall assume throughout the paper that bidder 0 always bids truthfully. Our equilibrium characterization is therefore focused on finding the bidding strategy of the *uninformed* bidders.

Additional discussion on our model. When studying information asymmetry in auctions, it is common to consider correlated bidder values since when values are independent, one bidder's information would not affect any other bidders' valuations. In this literature, the valuation model with a common and a private component is prevalent (Milgrom and Weber 1982; Abraham et al. 2013; Li and Das 2019). Moreover, it also strictly generalizes the classic common-value auction model (when $V_i \equiv 0, \forall i$) originating from the seminal work of Wilson (1966; 1967). Most related to our specific model is that of (Arnosti, Beck, and Milgrom 2016), motivated by online advertising, in which C captures the attributes of the auctioned user impression that are valuable to all advertisers (i.e., bidders), V_i captures bidder i's private value per impression, and they are independent. In another motivating domain of art auction, C captures the resale value of the auctioned artwork and V_i captures bidder i's personal preferences about the artwork (Goetzmann and Spiegel 1995).

Theme and structure of the remainder. A main theme of this paper is to study whether bidder 0's more accurate value estimation would necessarily benefit herself and harm other bidders. To do so, we first characterize the equilibrium of the game in Sections 3 and 4 for the cases of two bidders and many bidders, respectively. Section 5 presents our findings and insights by analyzing the equilibrium.

3 Equilibrium Analysis: Two-Bidder Cases

To understand bidder payoffs at equilibrium, we must be able to characterize the equilibrium in the first place. In this section, we focus on the setting with two bidders, i.e., bidder 0 and bidder 1, and provide an explicit equilibrium characterization under reasonable distribution assumptions.

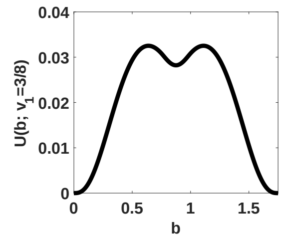
Since bidder 0 always bids her true value, we only need to characterize bidder 1's equilibrium bidding function, denoted as $b^*(v_1)$, which maps her realized private value v_1 to the optimal bid $b^*(v_1)$. Particularly, given any private value $v_1 \in [l_1, u_1]$, the equilibrium bid of bidder 1 is the b^* that maximizes her expected utility, as a function of v_1 and b, expressed as follows:

$$
\begin{aligned}
& U(b; v_1) \\
= \;& \mathop{\mathbf{E}}_{C,V_0}[v_1 + C - (V_0 + C) | b \geq V_0 + C] \cdot \mathbf{Pr}(b \geq V_0 + C) \\
= \;& \int_{v_0 + c \leq b} (v_1 - v_0) f_0(v_0) g(c) dv_0 dc \\
= \;& \int_0^b \int_0^{b - v_0} (v_1 - v_0) f_0(v_0) g(c) dc\, dv_0 \\
= \;& \int_0^b (v_1 - v_0) f_0(v_0) G(b - v_0) dv_0 \qquad (1)
\end{aligned}
$$

Since V_0 has support $[l_0, u_0]$. If $v_1 \geq u_0$, for any v_0 we have $(v_1 - v_0) f_0(v_0) G(b - v_0) \geq 0$. Therefore, $U(b; v_1)$ will be maximized by any $b \geq u_0 + u_c$. That is, bidder 1 simply bids high enough to secure a winning position. On the other hand, if $v_1 \leq l_0$, bidder 1 will bid as low as possible as any winning will result in negative utility. Therefore, in the rest of this section we focus on the non-trivial case $v_1 \in (l_0, u_0)$.

By definition, the equilibrium bidding function $b^*(v_1) = \mathrm{argmax}_b U(b; v_1)$ for any v_1. Unfortunately, we show that the optimal bid b^* generally does not admit tractable structure and may have multiple maxima because the function $U(b; v_1)$ as a function of b is non-convex in general for a fixed v_1. This is illustrated in the following example.

Example 1. *Consider the problem with* $V_0 \sim \mathcal{U}\left([0, \frac{3}{4}]\right)$ *where* $\mathcal{U}(I)$ *denotes the* uniform distribution *over set* I. *The common component* $C \sim \frac{1}{2}(Beta(2,7) + Beta(7,2))$, *i.e., the uniform mixture of* $Beta(2,7)$ *and* $Beta(7,2)$. *Simple calculation shows that* $g(c) = 28[c(1-c)^6 + c^6(1-c)]$ *for all* $c \in [0,1]$. *In this case, the explicit expression of bidder* 1*'s expected utility* $U(b; v_1)$*, as a function of* b*, is already rather complicated and thus omitted here. The following figure plots* $U(b; v_1)$ *for* $v_1 = 3/8$. *As we can see,* $U(b; v_1)$ *is neither convex nor concave, and has multiple local maxima.*



Example 1 reveals some intricacies in characterizing bidder 1's equilibrium bidding strategy due to multiple optimal bids. Generally, the optimal bidding function does not admit an explicit characterization. Our main result in this section is to identify a fairly general condition, under which bidder 1's optimal bidding strategy can be explicitly characterized and also easily computed.

Theorem 1. *If* $g(c)$ *is* log-concave[2], *then* $b^*(v_1)$ *is bidder* 1*'s equilibrium (i.e., optimal) bid for* v_1 if and only if $b^*(v_1)$ *satisfies equation* $v_1 = \mathbb{E}(V_0 | V_0 + C = b^*(v_1))$. *Moreover,* $\mathbb{E}(V_0 | V_0 + C = b)$ *is non-decreasing in* b*, therefore* $b^*(v_1)$ *can be efficiently computed for each* v_1 *via* binary search.

Theorem 1 shows that the bid b^* is optimal for bidder 1 if and only if conditioning on bidder 0 bidding b^*, the expected private value of bidder 0 equals v_1. It provides an explicit characterization of *all* the equilibrium bids for bidder 1. Besides revealing structural insights, this characterization also implies an efficient binary-search algorithm due to monotonicity of $\mathbb{E}(V_0 | V_0 + C = b)$ as a function of b.

The proof of Theorem 1 is rather involved and is deferred to Appendix A. Key to our proof is to rule out all bad situations similar to Example 1, for which a characterization of the optimal bid is challenging. To overcome this issue, we leverage the log-concavity assumption to prove the following crucial property of random variables, which may be of independent interest.

Lemma 2. *If* $g(c)$ *is* log-concave, *then* $\mathbf{E}(V_0 | V_0 + C = b)$ *is non-decreasing in* b.

This property allows us to show that bidder 1's utility function must increase first and then decrease. This allows us to characterize the optimal bid as the one with derivative equaling 0, leading to the condition in Theorem 1.

We remark that the log-concavity assumption in Theorem 1 is widely adopted for modeling value distributions (Bagnoli and Bergstrom 2006). In fact, many commonly used distributions — e.g., normal, exponential, logistic, Laplace and uniform distributions over a convex set — are all log-concave. It is easy to verify that the density function $g(c)$ in Example 1 is *not* log-concave. The following example is an application of Theorem 1.

[2] A function g is log-concave if $\log(g)$ is a concave function.

Example 2. *Consider the case where V_0 is drawn from the log-concave distribution $\mathcal{U}([0,\alpha])$ for some $\alpha \in (0,1]$ and $C \sim \mathcal{U}([0,1])$. Applying Theorem 1, we have*

$$b^*(v_1) = \begin{cases} 2v_1 & v_1 \in [0, \alpha/2); \\ \textit{any bid in } [\alpha, 1] & v_1 = \alpha/2; \\ 2v_1 + 1 - \alpha & v_1 \in (\alpha/2, \alpha]. \end{cases}$$

Example 2 already reveals interesting insights regarding bidding behaviors under information asymmetry. When $v_1 < \frac{1}{2}\alpha$, bidder 1 bids $2v_1$ which is less than $v_1 + \mathbf{E}(C)$, despite knowing that the common value C has high expected value $\mathbf{E}(C) = 0.5$. On the other hand, when $v_1 > \frac{1}{2}\alpha$, bidder 1 dramatically increases her bid to $2v_1 + 1 - \alpha > 1$, which is much larger than both her private component v_1 and the expectation of the common value. That is, *bidder* 1 *bids conservatively when her private value is small but bids aggressively otherwise*. It turns out that this intuition can be made precise, as shown in the following proposition.

Proposition 1. *If $f_0(v_0)$ and $g(c)$ are both log-concave, then there exists a **critical value** $t \in [l_1, u_1]$, such that $b^*(v_1) \le v_1 + \mathbf{E}(C)$ for any $v_1 < t$ and $b^*(v_1) \ge v_1 + \mathbf{E}(C)$ for any $v_1 > t$.*

Notice that if bidder 0 cannot observe the realized common value c either (so both bidders have equal information regarding C), then bidder 1's equilibrium bid will be $v_1 + \mathbf{E}(C)$. Proposition 1 shows that the *critical value* t is precisely the value which determines whether bidder 1 will bid more conservatively or aggressively, compared to the case where both bidders do not observe C.

Proof of Proposition 1. We start with a corollary of Lemma 2. In particular, by exchanging the role of V_0 and C, we have the following conclusion.

Corollary 1. *If $f_0(v)$ is* log-concave*, then $\mathbf{E}(C|V_0+C=b)$ is non-decreasing in b.*

By Theorem 1, we know that the optimal bid b^* for v_1 satisfies $v_1 = \mathbf{E}(V_0|V_0 + C = b^*) = b^* - \mathbf{E}(C|V_0 + C = b^*)$. So $b^* = v_1 + \mathbf{E}(C|V_0 + C = b^*)$. Therefore,

$$b^* - (v_1 + \mathbf{E}(C)) = \mathbf{E}(C|V_0 + C = b^*) - \mathbf{E}(C). \quad (2)$$

Observe that $\lim_{\epsilon\to 0^+} b^*(l_0 + \epsilon) = l_0 + l_c$, i.e., if bidder 1's value equals the minimum possible bidder 0 value l_0, the $l_0 = \mathbf{E}(V_0|V_0+C = b^*(l_0))$ condition implies that $b^*(l_0) = l_0 + l_c$. Thus $\lim_{\epsilon\to 0^+} \mathbf{E}(C|V_0 + C = b^*(l_0 + \epsilon)) = l_c < \mathbf{E}(C)$. Similarly, $\lim_{\epsilon\to 0^+} \mathbf{E}(C|V_0 + C = b^*(u_0 - \epsilon)) = u_c > \mathbf{E}(C)$. By Corollary 1, if we increase v_1 from l_0 to u_0, the sign of $b^*(v_1) - (v_1 + \mathbf{E}(C))$ will change from being negative to being positive. There must exist a critical point t such that $b^*(t) = t + \mathbf{E}(C)$, while $b^*(v_1) \le v_1 + \mathbf{E}(C)$ for all $v_1 < t$ and $b^*(v_1) \ge v_1 + \mathbf{E}(C)$ for all $v_1 > t$.

□

4 Equilibria Analysis: Many-Bidder Cases

In this section, we consider the case with $n \ge 2$ uninformed bidders. Like most previous works (Milgrom and Weber 1982; Engelbrecht-Wiggans, Milgrom, and Weber 1983; Arnosti, Beck, and Milgrom 2016), we focus on *symmetric* uninformed bidders — i.e., V_i's are independent and identically distributed (i.i.d.). They are drawn from the same continuos distribution $f(v)$ supported on the entire interval $[l, u]$ for all bidder $i = 1, 2, ..., n$. However, we do allow $f_0 \ne f$ to be different. Note that if bidder $i (> 0)$ has a value v less than l_0, he would always prefer to not win the auction and thus bid 0. Therefore, we shall w.l.o.g. assume $l \ge l_0$. Also similar to these previous work, we focus on *symmetric equilibria* among uninformed bidders. By "symmetric equilibrium" we mean the same bidding strategy $b^*(v)$ for any *uninformed* bidder $i > 0$.[3]

Before characterizing the equilibrium, we first show a useful property of any symmetric equilibrium. That is, the equilibrium bid strictly increases in the private value v. Its proof is technical and deferred to Appendix B.

Lemma 3. ***(Strict Bid Monotonicity)*** *Any symmetric equilibrium $b^*(v)$ is* strictly *increasing in the private value v. In particular, for any $l_0 < v < v'$, $b^*(v) < b^*(v')$.*

Lemma 3 implies some structural properties of the equilibrium bidding function $b^*(v)$. First, $b^*(v)$ is almost everywhere differentiable, since it is a bounded strictly increasing function. Moreover, the distribution of the bid $b^*(V)$, denoted as $P(b)$, is *atomless* since $b^*(v)$ is strictly increasing. Next, we characterize the symmetric equilibrium by providing a first-order characterization of the bid distribution $P(b)$. We can easily recover $b^*(v)$ from $P(b)$ since $F(v) = P(b^*(v))$, i.e., $b^*(v) = P^{-1}(F(v))$.

Theorem 2. *Suppose a symmetric equilibrium exists, then it is the unique symmetric equilibrium. Moreover, the bid distribution $P(b)$ of this symmetric equilibrium is characterized by the following first-order differential equation:*

$$P' = -\frac{P\int_0^b (F^{-1}(P) - x) f_0(x) g(b-x)\,dx}{(n-1)\int_0^b (F^{-1}(P) - x) F_0(x) g(b-x)\,dx}, \quad (3)$$

for any $b \in [\min^*, u_c + u]$ with boundary condition $P(\min^*) = 0$, where $\min^*$ is the maximum b satisfying $b = l + \mathbf{E}(C|C + V_0 \le b)$.

A proof sketch of Theorem 2 goes as follows. We start by characterizing the equilibrium utility of any uninformed bidder, as a function of the *functional* $P(b)$. By definition of equilibrium, we know that the bid b must be the optimal bid for the bidder value $v = F^{-1}(P(b))$ for any b, implying that the derivative of the revenue, viewed as a function of b for the corresponding given v, must equal 0. This gives rise to a differential equation as stated in Equation (3). The formal proof is technical, and is deferred to Appendix C, in which we also provide a numerical algorithm for computing an approximation of the distribution $P(b)$ based on the finite element method.

In the next section, we shall derive a closed form solution to the above differential equation for uniformly random distributions. Before concluding this section, we briefly mention how Equation (3) is related to our equilibrium characterization for two bidder cases ($n = 1$) as in Theorem 1.

[3]Recall that bidder 0 has more accurate estimation of C and always bids truthfully due to Lemma 1.

Since the value v and the corresponding equilibrium bid b satisfies $v = F^{-1}(P(b))$, when $n = 1$, we must have the numerator of the RHS of Equation (3) equal to 0, in order to yield bounded P'. With $v = F^{-1}(P(b))$, this exactly implies $\int_0^b (v-x)f_0(x)g(b-x)\,dx = 0$, or equivalently,

$$v = \frac{\int_0^b xf_0(x)g(b-x)\,dx}{\int_0^b f_0(x)g(b-x)\,dx} = \mathbf{E}(V_0|V_0 + C = b)$$

The above condition is precisely the characterization of Theorem 1.

5 Strange Roles of Information Asymmetry

Armed with the equilibrium characterization in Section 3 and 4, we are now ready to study how information asymmetry affects bidders' utilities in a second price auction. We shall compare the following two different situations:

- **Asymmetric World**: only bidder 0 observes realized c (our current model).
- **Symmetric World**: none of the bidders observe the realized c.

5.1 The Harm of Overtly Possessing Information

At a first glance, one may think that bidder 0, as the only informed bidder, would always get higher utility in the asymmetric world. Our first surprising finding is that this is not the case. In fact, more accurate value estimation may be harmful for all possible bidder value realizations.

Proposition 2. *Suppose all bidders know that they are in the asymmetric world. There are instances in which bidder* 0 *in this asymmetric world has* strictly less *utility than her utility in the symmetric world for* any *possible bidder* 0 *type. That is, overtly possessing information can be harmful to the informed bidder* 0.

Proof. Our proof constructs such an instance with two bidders, and then utilize our equilibrium characterization in Theorem 1 and Proposition 1. Our main insight is that if the uninformed bidder's valuation is close to 0 with high probability, then the uninformed bidder will tend to bid aggressively under information asymmetry. Specifically, the critical value in Proposition 1 would be very small. So that with almost probability 1, bidder 1 will over bid compared to the case when bidder 0 is not informed and thus causes utility decrease for bidder 0. Next, we provide a formal construction.

Consider the case where the valuation density of bidder 0 is $\frac{\lambda e^{-\lambda v_0}}{1-e^{-\lambda}}$ for $v_0 \in [0, 1]$, i.e., a truncated exponential distribution. We will mostly think of λ as large enough. The density function for bonus value c and bidder 1's valuation is the same — the uniform distribution on $[0, 1]$. Notice that these two distributions are both log-concave.

By Proposition 1, we know that there exists a critical value $\widehat{v}$, such that the uninformed bidder bids strictly more than $v_0 + \mathbf{E}(C) = v_0 + \frac{1}{2}$. We now show that in the constructed example, $\widehat{v} < \frac{1}{\lambda}$. In fact, as $\lambda \to \infty$, bidder 1 will almost always bid $v_1 + 1$.

Specifically, based on Theorem 1, we can derive the relation between value v_1 and its corresponding bid b_1 for bidder 1, as follows:

$$v_1 = \frac{1}{\lambda} - \frac{b_1}{e^{\lambda b_1} - 1},\ \forall b_1 \in [0, 1] \tag{4}$$

$$v_1 = \frac{1}{\lambda} + \frac{(b_1 - 1)e^{\lambda(2-b_1)} - 1}{e^{\lambda(2-b_1)} - 1},\ \forall b_1 \in [1, 2] \tag{5}$$

By Equation (4) we know that, for any $b_1 \leq 1$, we must have $v_1 < \frac{1}{\lambda}$. Therefore, for any $v_1 \geq \frac{1}{\lambda}$, its bid b_1 must be decided by Equation (5). As $\lambda \to \infty$, the right hand side of Equation (5) tends to $b_1 - 1$. This induces that bidder 1 tends to bid $v_1 + 1$ for all $v_1 \geq \frac{1}{\lambda}$ as $\lambda \to \infty$. In addition, the probability that $v_1 \leq \frac{1}{\lambda}$ also tends to 0, since v_1 is drawn from a uniform distribution over $[0, 1]$. Therefore, as $\lambda \to \infty$, bidder 1 almost surely bids v_1+1 and the utility of bidder 0 tends to

$$\int_{v_0+c\geq v_1+1} (v_0 + c - v_1 - 1)f(v_1)g(c)\,dv_1\,dc = \frac{1}{6}v_0^3$$

Notice that if bidder 0 is not informed, both bidders will "truthfully" bid $v_i + \frac{1}{2}$, in which case bidder 0, with value v_0, gets expected utility $\int_{v_0\geq v_1}(v_0 - v_1)f(v_1)\,dv_1 = \frac{1}{2}v_0^2$, which is greater than $\frac{1}{6}v_0^3$ for any $v_0 \in [0, 1]$. □

Intuitively, the proof of Proposition 2 illustrates that when a "weak" bidder (with generally smaller private value) is *publicly* known to be informed, a "strong" bidder (with generally larger private value) will tend to bid more aggressively, and this may cause utility loss to the weak bidder.

Notably, it is important to assume all bidders know that they are in the asymmetric world, i.e., bidder 0 *overtly* possesses information. A simple observation reveals that if bidder 0 is *covertly* informed,[4] her utility will always increase. This is because when she is covertly informed, bidding the true value $v_0 + c$ dominates the strategy of bidding $v_0 + \mathbf{E}(C)$, under which she will already get utility that is equal to her utility when she is uninformed. This suggests that any bidder may prefer covertly acquiring information, especially for weak bidders.

5.2 Possible Blessing of Less Information

Next, we continue to explore uninformed bidders' utilities. A natural conjecture would be that uninformed bidders may always prefer the symmetric world, i.e., perhaps the asymmetric world always leads to utility decrease for the *disadvantaged* uninformed bidders.

Unsurprisingly, this can indeed happen. For instance, consider previous two-bidder Example 2 with $\alpha = 1$, i.e., $f_0(x)$ and $g(c)$ are both uniform distribution on $[0, 1]$. We have shown that in the asymmetric world, bidder 1's bidding function is $b(v_1) = 2v_1$ for all $v_1 \in [0, 1]$. Basic algebraic cal-

[4]More precisely, all bidders thought that they are in the symmetric world but bidder 0 covertly gets information to make it actually the asymmetric world .

culation yields that in the asymmetric world, bidder 1's expected utility is as follows:[5]

$$\text{ExpU}_1 = \begin{cases} \frac{2}{3}v_1^3 & v_1 \in [0, 0.5]; \\ -\frac{2}{3}v_1^3 + 2v_1^2 - v_1 + \frac{1}{6} & v_1 \in [0.5, 1]. \end{cases}$$

On the other hand, in the symmetric world, it is straightforward to verify that bidder 1's equilibrium bid is $v_1 + 0.5$, resulting in expected utility $v_1^2/2$, which turns out to be larger than the above utility in the asymmetric world for *any* v_1.

What is surprising, however, is that the above "disadvantaged" situation for uninformed bidders is not always harmful. There are situations where the uninformed bidders can do *better* in the asymmetric world.

Proposition 3. *There are instances in which an uninformed bidder of certain type derives* strictly more *utility in the asymmetric world than her utility in the corresponding symmetric world.*

Proof. We consider the setting with $n+1$ bidders and all distributions (f_i, g) are the uniform distribution over $[0,1]$. Due to valuation symmetry among uninformed bidders, let $i(>0)$ denote a generic bidder of our interest and v denote i'th private value.

We start by analyzing the simpler case, i.e., the symmetric world. In this case, bidder 0 does not observe c; it is easy to see that the bidder i bids $v + \frac{1}{2}$ at the unique dominant-strategy equilibrium. Therefore, the expected utility for bidder i with basic value v is $\int_0^v (v-x)nx^{n-1}dx = \frac{1}{n+1}v^{n+1}$.

More intricate is the equilibrium analysis in the asymmetric world in which bidder 0 observes c. We will use the equilibrium characterization in Theorem 2, and first pin down the boundary condition. Since $l = 0$, it is easy to see that the only b that satisfies $b = \mathbf{E}(C|C + V_0 \le b)$ is $b = 0$. Therefore, the bid for the smallest value of bidder i is 0.

Next, we analyze bidder i's bidding function $b(v)$. Since $V \sim U[0,1]$, b has a corresponding distribution, with randomness inherited from V, with CDF $P(b)$. Equation (3) is a differential equation that characterizes $P(b)$. We now solve for $P(b)$ for the special case that f_i, g are all uniform distribution over $[0,1]$. Note that $P(b) = F_i(v) = v$, we thus have

$$F_0^{-1}(P(b)) = F_0^{-1}(F_i(v)) = F_0^{-1}(F_0(v)) = v = P(b)$$

Plugging the above equality into Equation (3), and utilize the fact that f_0, g are both uniform distribution over $[0,1]$, we have

$$P'(b) = -\frac{P(b)[P(b)b - b^2/2]}{(n-1)[P(b)s(b) - t(b)]} \qquad (6)$$

with boundary condition $P(0) = 0$, where function $s(b) = \int_0^b F_0(x)g(b-x)dx$ is

$$s(b) = \begin{cases} \frac{1}{2}b^2 & b \in [0,1]; \\ -\frac{1}{2}b^2 + 2b - 1 & b > 1 \end{cases}$$

and function $t(b) = \int_0^b xF_0(x)g(b-x)dx$ is

$$t(b) = \begin{cases} \frac{1}{3}b^3 & b \in [0,1]; \\ \frac{1}{6}(-2b^3 + 9b^2 - 6b + 1) & b > 1 \end{cases}$$

It turns out that when $b \le 1$, differential inequality has a closed form unique solution $P(b) = \frac{2n+1}{3n+3}b$, which can be verified simply by plugging it into the function. When $b > 1$, it appears intractable to have a closed form solution due to much more intricate format of function $s(b), t(b)$. However, our argument next will only need to focus on $b \le 1$ – or equivalently, $v \le \frac{2n+1}{3n+3}$ – for the generic bidder i.

The above derivation shows that at the symmetric equilibrium the uninformed bidder i's bidding strategy is $\frac{3n+3}{2n+1}v$ for private value v satisfying $v \le \frac{2n+1}{3n+3}$ (i.e., the bid is at most 1). We are now ready to compute bidder i's utility at such a private value v. For convenience, let $\alpha = \frac{3n+3}{2n+1}$, so $b(v) = \alpha v$. We then have

$$\begin{aligned}
& U(v) \\
= & \int_{v \ge u, \alpha v \ge v_0 + c} (v + c - \max\{v_0 + c, \alpha u\}) \\
& \quad \times (n-1)u^{n-2} du\, dv_0\, dc \\
= & \int_{v \ge u \ge \frac{1}{\alpha}(v_0 + c)} (v + c - \alpha u)(n-1)u^{n-2} du\, dv_0\, dc \\
& + \int_{v \ge \frac{1}{\alpha}(v_0+c) \ge u} (v - v_0)(n-1)u^{n-2} du\, dv_0\, dc \\
= & \int_{v \ge u \ge \frac{1}{\alpha}c} (v + c - \alpha u)(n-1)u^{n-2}(\alpha u - c) dc\, du \\
& + \int_{\alpha v \ge v_0 + c} (v - v_0)(\frac{v_0 + c}{\alpha})^{n-1} dc\, dv_0 \\
= & \int_{v \ge u} (\frac{v(u\alpha)^2}{2} - \frac{(\alpha u)^3}{3})(n-1)u^{n-2} du \\
& + \int_{\alpha v \ge v_0} (v - v_0)\frac{\alpha}{n}(v^n - (\frac{v_0}{\alpha})^n) dv_0 \\
= & \left[\frac{\alpha^2(n-1)}{2(n+1)} - \frac{\alpha^3(n-1)}{3(n+2)}\right] v^{n+1} \\
& + \frac{\alpha v^{n+1}}{n}\left[\alpha - \frac{\alpha^2}{2} - \frac{\alpha}{n+1} + \frac{\alpha^2}{n+2}\right]
\end{aligned}$$

We now compare $U(v)$ with the bidder's utility $\frac{1}{n+1}v^{n+1}$ in the symmetric world at $v = \frac{1}{\alpha}$. Considering sufficiently large n and omitting $o(1)$ terms, we have

$$\begin{aligned}
& \frac{U(v) - \frac{1}{n+1}v^{n+1}}{v^{n+1}} \\
= & \frac{\alpha^2(n-1)}{2(n+1)} - \frac{\alpha^3(n-1)}{3(n+2)} + \frac{\alpha^2 - \alpha^3/2}{n} - O(\frac{1}{n^2}) - \frac{1}{n+1} \\
= & \alpha^2[\frac{1}{2n} + \frac{\alpha^2 - \alpha^3/2}{n}] - O(\frac{1}{n^2}) - \frac{1}{n+1}
\end{aligned}$$

When n is large, $\alpha \approx \frac{3}{2}$, so the above quantity tends to $\frac{9}{16n} > 0$, omitting $O(\frac{1}{n^2})$ term. Therefore, $U(v) >$

[5]The different utility formats come from different integral bounds when $b(v_1) = 2v_1$ is larger or smaller than 1.

$\frac{1}{n+1}v^{n+1}$. In other words, an uninformed bidder with value $v = 1/\alpha$ gets strictly larger utility in the asymmetric world than her utility in the symmetric world. □

Our next result shows that the above phenomenon may be due to the intense competition among many bidders, under which: (1) each bidder derives little utility in the symmetric world; whereas (2) more conservative bidding among uninformed bidders in the asymmetric world helps them to increase utility when their private values are high. It turns out that with two bidders, the uninformed bidder will always be worse off in the asymmetric world.

Proposition 4. *When there are two bidders (one informed and one uninformed), the utility of the uninformed bidder* 1 *in the asymmetric world is always less than her utility in the symmetric world, for any private value* v_1.

Proof. We have

$$\begin{aligned} & U(b) \\ = \ & \mathbf{E}([v_1 + C] - [V_0 + C] | V_0 + C \leq b) \times P(V_0 + C \leq b) \\ = \ & \int_0^b (v_1 - v_0) f(v_0) G(b - v_0) dv_0 \\ = \ & \int_0^{v_1} (v_1 - v_0) f(v_0) G(b - v_0) dv_0 \\ & \quad + \int_{v_1}^{b} (v_1 - v_0) f(v_0) G(b - v_0) dv_0 \\ \leq \ & \int_0^{v_1} (v_1 - v_0) f(v_0) dv_0 + 0 \end{aligned}$$

The last term $\int_0^{v_1} (v_1 - v_0) f(v_0) dv_0$ is precisely bidder 1's utility in the symmetric world. □

5.3 Welfare, Revenue, and Failure of the Linkage Principle

Finally, we make a few simple observations about the effect of information asymmetry on revenue and welfare.

Observation 1. *The auction's welfare in the asymmetric world is always weakly less than that in the symmetric world.*

Proof. This is because if bidder 0 was not informed as in the symmetric world, the second price auction is socially optimal — it always allocates the item to the bidder with higher private values, which is also the bidder with the highest total value since the common component is shared among bidders. □

Example 3 (Winner's curse and revenue collapse)**.** *Consider the degenerate case where* $V_i \equiv 0$ *and bidders have only the common value* C*, drawn from the uniform distribution on* $[0, 1]$. *In the asymmetric world, bidder* 0 *always bids his true value* c *whereas every other bidder's equilibrium bid is always* 0. *This is due to the* winner's curse — *had any bidder* $i (> 0)$ *won, she must have bid higher than bidder* 0*'s bid which is the true value* c *for everyone. The revenue in this asymmetric world is* 0, *while the revenue in the symmetric world is* $\mathbf{E}(C) = 0.5$.

In auction theory, a seminal result of (Milgrom and Weber 1982) is the well-known *linkage principle*. That is, when bidders are symmetric and values are associated, revealing private information of the auctioneer (if she has any) always increases revenue in most commonly seen auctions, including the second-price auction. In other words, *transparency always favors revenue*.

Recent result by (Syrgkanis, Kempe, and Tardos 2015) shows that this principle fails in setting with two asymmetrically informed bidders, however their proof is involved due to intricate equilibrium analysis under asymmetric information. Next, we use a much simpler example in our setting to give another piece of evidence for the failure of the linkage principle, also with two asymmetrically informed bidders. The key intuition is that the uninformed bidder tends to bid aggressively, which leads to higher revenue for the auctioneer, compared to the situation in which bidder 1 knows exactly the realized c. Therefore, the auctioneer may prefers (some) less informed bidders.

Example 2 Cont'd. (Failure of the Linkage Principle) *Recall the Example 2 setup with two bidders. Now consider* f_1 *as a point-distribution with all point mass at* $v_1 = 0.8\alpha$. *Bidder 1's equilibrium bid is (deterministically)* $b_1 = 2v_1 + 1 - \alpha = 1 + 0.6\alpha$. *Given full information, bidder 0's dominant strategy is to bid true value* $b_0 = v_0 + c$. *Note that* $\mathbf{Pr}(V_0 + C \geq 1 + 0.6\alpha) = \mathbf{Pr}(1 - C \leq V_0 - 0.6\alpha) = \int_{0.6\alpha}^{\alpha} (v_0 - 0.6\alpha) dF_0(v_0) = 0.08\alpha$. *Therefore,*

$$\begin{aligned} Rev &= \int_{v_0 + c \leq 1 + 0.6\alpha} (v_0 + c) f_0(v_0) g(c) dv_0 dc \\ & \quad + (1 + 0.6\alpha) \cdot 0.08\alpha \\ &= 0.5 + 0.5\alpha - 0.0107\alpha^2 \end{aligned}$$

Now suppose the auctioneer has private information and can observe realized c. If the auctioneer reveals c to bidder 1, then both bidders will bid truthfully, leading to revenue

$$\mathop{\mathbf{E}}_{V_0, C} \min\{V_0 + C, 0.8\alpha + C\} = \mathop{\mathbf{E}}_{V_0} \min\{V_0, 0.8\alpha\} + \mathbf{E}[C],$$

which equals $0.5 + 0.48\alpha$ *after basic calculation. Therefore, for any* $\alpha \leq 1$, *the revenue under no information is larger than that under full information, leading to the failure of the linkage principle.*

6 Conclusion

In this paper, we study how information asymmetry affects the equilibrium of a second price auction. The key message revealed from our results is that in strategic interactions as the second-price auction, competition among bidders makes the role of asymmetric information intricate. For instance, less information could be beneficial whereas more information could be harmful. Our work thus leads to new insights about bidders' behaviors, and shed light on the situations under which bidders may or may not want to obtain fine-grained information about the auctioned item. This leaves many open directions for future research, e.g., understanding the role of information asymmetry in other auction models with different bidder value models and more generally, in other strategic games.

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Appendix for "The Strange Role of Information Asymmetry in Auctions"

A Proof of Theorem 1 and Proposition 1

Our proof of Theorem 1 crucially relies on the following notion of *affiliation* between two random variables, which is introduced by Karlin and Rubin (1956).

Definition 1 (Affiliation (Karlin and Rubin 1956))**.** *Let (V, W) be a two-variate random variable and $\pi(v, w)$ be the density function of (V, W). V and W are* affiliated *if for all $v < v'$ and $w < w'$,*

$$\pi(v, w)\pi(v', w') \geq \pi(v, w')\pi(v', w) \tag{7}$$

We will think of V as the private value in a bidder's valuation and $W = V + C$ as the bidder's total valuation. If V and W are *affiliated*, intuitively it means that it is more likely that a bidder's private value and total valuation are both high or low than that one of them is high while another is low. Interestingly, it turns out that the affiliation relation between V_i and $W_i = V_i + C$ is equivalent to the log-concavity of $g(c)$, and surprisingly, does not depend on the distribution of V_i. This also illustrates the rationale of our log-concavity assumption of $g(c)$.

Lemma 4. *For any $i \in [n]$, V_i and $W_i = V_i + C$ are affiliated* if and only if *$g(c)$ is* log-concave.

Proof. In this proof, we will drop the subscript i and work with $v, w, \pi(v, w)$ for notational convenience. Observe that $\pi(v, w) = f(v)g(w - v)$ is the density function of the two-variate random variable (V, W)[6]. Given any $v < v'$ and $w < w'$, w.l.o.g., assume v, v' all have *strictly* positive probability density, since otherwise the affiliation equation is naturally satisfied. Equation (7) then induces

$$\begin{aligned}
& \pi(v, w)\pi(v', w') \geq \pi(v, w')\pi(v', w) \\
\Leftrightarrow \quad & f(v)g(w - v)f(v')g(w' - v') \geq f(v)g(w' - v)f(v')g(w - v') \\
\Leftrightarrow \quad & g(w - v)g(w' - v') \geq g(w' - v)g(w - v')
\end{aligned}$$

Let $c = w - v$, $c' = w' - v'$, $s = w' - v$ and $s' = w - v'$. Therefore, we have $c + c' = s + s'$. Moreover, $v < v'$ and $w < w'$ imply that $s > \max(c, c')$ and $s' < \min(c, c')$. We only consider the case $c \geq c'$ since the case $c < c'$ is similar. When $c \geq c'$, we have $s > c \geq c' > s'$ and $s - c = c' - s' > 0$. Using these facts, we have

$$\begin{aligned}
& g(w - v)g(w' - v') \geq g(w' - v)g(w - v') \\
\Leftrightarrow \quad & g(c)g(c') \geq g(s)g(s') \\
\Leftrightarrow \quad & \log g(c) + \log g(c') \geq \log g(s) + \log g(s') \\
\Leftrightarrow \quad & \frac{\log g(c') - \log g(s')}{c' - s'} \geq \frac{\log g(s) - \log g(c)}{s - c}
\end{aligned} \tag{8}$$

for all $s > c \geq c' > s'$. However, since v, v', w, w' can take any value in their supports, the values of s, c, c', s' can therefore span the whole support of C. As a result, Inequality (8) is equivalent to that $\log g(c)$ is concave, i.e., $g(c)$ is log-concave. Therefore, V and $W = V + C$ are affiliated if and only if $g(c)$ is log-concave. □

We now prove the monotonicity of $\mathbf{E}(V_0|V_0 + C = b)$ as a function of b.[7] At a high level, utilizing the affiliation relation between V_0 and W_0, we will show the stochastic order of the random variable V_0 conditioned on $W_0 = b$ for different b's. We then prove the monotonicity of the conditional expectations based on the stochastic order.

Lemma 5. *[Restatement of Lemma 2] If $g(c)$ is* log-concave*, then $\mathbf{E}(V_0|V_0 + C = b)$ is non-decreasing in b.*

Proof. With some abuse of notation, let V_b denote the random variable V_0 after conditioning on $V_0 + C = b$, i.e., $V_b = (V_0|V_0 + C = b)$. We now prove that V_b is *stochastically increasing* in b. That is,

$$\mathbf{Pr}(V_b \leq t) \geq \mathbf{Pr}(V_{b'} \leq t), \qquad \forall\, b' > b, t \in \mathbb{R}. \tag{9}$$

Let $\pi(v_0, w_0) = f(v_0)g(w_0 - v_0)$ be the density function of the two-variate random variable (V_0, W_0), $\pi(w_0)$ be the marginal distribution of W_0, and $\pi(v_0|w_0 = b)$ be the conditional distribution. Note that, $\pi(v_0|w_0 = b)$ is precisely the distribution of

[6] Recall that V and C are independent.

[7] Note that this is not true in general. See, e.g., the setting of Example 1.

the random variable V_b. We will use Π to denote the CDF of these distributions. Since $g(c)$ is log-concave, by Lemma 4 we know that V_0 and W_0 are affiliated. Therefore, for all $v < v'$ and $b < b'$, we have

$$
\begin{aligned}
& \pi(v,b)\pi(v',b') \geq \pi(v,b')\pi(v',b) \\
\Rightarrow \quad & \frac{\pi(v',b')\pi(v,b)}{\pi(b')\pi(b)} \geq \frac{\pi(v,b')\pi(v',b)}{\pi(b')\pi(b)} \\
\Rightarrow \quad & \pi(v'|b')\pi(v|b) \geq \pi(v|b')\pi(v'|b) \\
\Rightarrow \quad & \int_0^{v'} \pi(v'|b')\pi(v|b)dv \geq \int_0^{v'} \pi(v|b')\pi(v'|b)dv \\
\Rightarrow \quad & \pi(v'|b')\Pi(v'|b) \geq \Pi(v'|b')\pi(v'|b) \\
\Rightarrow \quad & \frac{\pi(v'|b')}{\Pi(v'|b')} \geq \frac{\pi(v'|b)}{\Pi(v'|b)} \\
\Rightarrow \quad & \int_t^{\infty} \frac{\pi(v'|b')}{\Pi(v'|b')}dv' \geq \int_t^{\infty} \frac{\pi(v'|b)}{\Pi(v'|b)}dv' \\
\Rightarrow \quad & 0 - \ln \Pi(t|b') \geq 0 - \ln \Pi(t|b) \\
\Rightarrow \quad & \Pi(t|b') \leq \Pi(t|b) \\
\Rightarrow \quad & \mathbf{Pr}(V_{b'} \leq t) \leq \mathbf{Pr}(V_b \leq t)
\end{aligned}
$$

for any $t \in \mathbb{R}$. This shows that V_b is stochastically increasing in b. Therefore, we must have $\mathbf{E}(V_b) \leq \mathbf{E}(V_{b'})$ for any $b' > b$. This is precisely $\mathbf{E}(V_0|V_0 + C = b) \leq \mathbf{E}(V_0|V_0 + C = b')$ for any $b' > b$, as desired. □

Proof of Theorem 1.

From Equation (1) we know that bidder 1's expected utility, given private value $v_1 \in (l_0, u_0)$, is

$$U(b; v_1) = \int_0^b (v_1 - v_0)f(v_0)G(b - v_0)dv_0$$

The equilibrium bid b^* must maximize the above utility. We now compute the derivative of $U(b; v_1)$ with respect to b for any $b \in (l_0 + l_c, u_0 + u_c)$, as follows:

$$
\begin{aligned}
U'(b; v_1) &= (v_1 - b)f(b)G(b - b) + \int_0^b (v_1 - v_0)f(v_0)g(b - v_0)dv_0 \\
&= v_1 \int_0^b f(v_0)g(b - v_0)dv_0 - \int_0^b v_0 f(v_0)g(b - v_0)dv_0 \\
&= \left[\int_0^b f(v_0)g(b - v_0)dv_0\right] \times \left[v_1 - \frac{\int_0^b v_0 f(v_0)g(b - v_0)dv_0}{\int_0^b f(v_0)g(b - v_0)dv_0}\right] \\
&= \left[\int_0^b f(v_0)g(b - v_0)dv_0\right] \times [v_1 - \mathbf{E}(V_0|V_0 + C = b)] \qquad (10)
\end{aligned}
$$

Note that for any $b \in (l_0 + l_c, u_0 + u_c)$, the event $V_0 + C = b$ has strictly positive probability density. Therefore, $\int_0^b f(v_0)g(b - v_0)dv_0 > 0$ and $\mathbf{E}(V_0|V_0 + C = b) = \frac{\int_0^b v_0 f(v_0)g(b-v_0)dv_0}{\int_0^b f(v_0)g(b-v_0)dv_0}$ is well-defined. As a result, by setting $U'(b) = 0$ we have

$$v_1 = \mathbf{E}(V_0|V_0 + C = b).$$

By Lemma **??**, we know that $\mathbf{E}(V_0|V_0 + C = b)$ is monotonically non-decreasing in b. Moreover, $\lim_{\epsilon \to 0^+} \mathbf{E}(V_0|V_0 + C = l_0 + l_c + \epsilon) = l_0 < v_1$ and $\lim_{\epsilon \to 0^+} \mathbf{E}(V_0|V_0 + C = u_0 + u_c - \epsilon) = u_0 > v_1$. By the continuity of $\mathbf{E}(V_0|V_0 + C = b)$ as a function of b, there must exist an interval $[b_1, b_2]$ (b_1 may equal b_2), such that $v_1 = \mathbf{E}(V_0|V_0 + C = b)$ for all $b \in [b_1, b_2]$, and $v_1 > \mathbf{E}(V_0|V_0 + C = b)$ if $b < b_1$, $v_1 < \mathbf{E}(V_0|V_0 + C < b)$ if $b > b_2$. In other words, $U(b; v_1)$ is strictly increasing for $b < b_1$, has a constant value for $b \in [b_1, b_2]$, and is strictly decreasing for $b > b_2$. Therefore it must achieve the global maximum at any $b^* \in [b_1, b_2]$, which satisfies $v_1 = \mathbf{E}(V_0|V_0 + C = b^*)$. This concludes our proof.

□

B Proof of Lemma 3

Let i denote a generic uninformed bidder with private value v_i. We first prove that his bid is *weakly* increasing in v_i. That is, for any $v_i < v_i'$, we have $b^*(v_i) \le b^*(v_i')$.

Let $Q(b)$ denote the probability that the highest bid among all the *uninformed* bidders, excluding bidder i, is b. We use b_i to denote bidder i's bid. With private value v_i, the expected utility of bidder i by bidding b_i is $U(b_i; v_i) = v_i \times P(b_i) - z(b_i)$ where $P(b_i) = Q(b_i)\big[\int_{b_i \ge v_0 + c} f(v_0) g(c) dv_0\, dc\big]$ is the probability that bidder i wins the auction by bidding b_i, and $z(b_i)$ is the expected payment which only depends on b_i. Observe that $P(b_i)$ is *strictly* increasing in b_i, because $Q(b_i)$ is non-decreasing in b_i while $\int_{b_i \ge v_0 + c} f(v_0) g(c)\, dv_0\, dc$ is strictly increasing in $b_i (\le u_0 + u_c)$.

We now prove the weak monotonicity of $b^*(v)$ by contradiction. If this is not true, then there exists $v_i < v_i'$ such that $b^*(v_i) > b^*(v_i')$. For notational convenience, let $b_i = b^*(v_i)$ and $b_i' = b^*(v_i')$. We must have $U(b_i; v_i) \ge U(b_i'; v_i)$ and $U(b_i'; v_i') \ge U(b_i; v_i')$ by the definition of equilibrium, therefore

$$
\begin{aligned}
& U(b_i; v_i) + U(b_i'; v_i') \ge U(b_i'; v_i) + U(b_i; v_i') \\
\Rightarrow\quad & v_i P(b_i) - z(b_i) + v_i' P(b_i') - z(b_i') \ge \\
& \quad v_i P(b_i') - z(b_i') + v_i' P(b_i) - z(b_i) \\
\Rightarrow\quad & v_i P(b_i) + v_i' P(b_i') \ge v_i P(b_i') + v_i' P(b_i) \\
\Rightarrow\quad & [v_i - v_i']\,[P(b_i) - P(b_i')] \ge 0
\end{aligned}
$$

On the other hand, $b_i > b_i'$ induces $P(b_i) > P(b_i')$ since $P(b_i)$ is *strictly* increasing. Therefore we should also have $[v_i - v_i']\,[P(b_i) - P(b_i')] < 0$, which yields a contradiction to the above Inequality.

We now argue that the symmetric equilibrium $b^*(v)$ is *strictly* increasing in v. Our proof is still by contradiction. If $b^*(v)$ is not strictly increasing, then there exist two valuations $v_l < v_r$, such that the equilibrium bid for v_l, v_r is the same (denoted as $\widetilde{b}$). Due to weak monotonicity, for all v in interval $[v_l, v_r]$, the equilibrium bid has to be the bid $\widetilde{b}$. This means, $U(b; v) = vP(b) + z(b)$, as a function of b, achieves maximum at $b = \widetilde{b}$ for any $v \in [v_l, v_r]$. We show that this is impossible.

Observe that the event that bid $\widetilde{b}$ is the winning bid and there is a tie among uninformed bidders has positive probability, because $\mathbf{Pr}(V_0 + C \le \widetilde{b}) > 0$ and $\mathbf{Pr}(v \in [v_l, v_r]) > 0$. We derive a contradiction by analyzing the competition among the *uninformed* bidders at the tie bid $\widetilde{b}$. In particular, we will next show that the equilibrium condition must imply $v + \mathbf{E}(C|V_0 + C \le \widetilde{b}) = \widetilde{b}$ for any $v \in [v_l, v_r]$, which however is not possible because $\widetilde{b} - \mathbf{E}(C|V_0 + C \le \widetilde{b})$ is a constant while $v \in [v_l, v_r]$ varies. This contradiction shows the strict monotonicity of $b^*(v)$.

When $\widetilde{b}$ is a tie, equilibrium condition implies that bidder i is not interested in bidding slightly higher to win the tie. This must be because her private value v_i plus the expected common value C, conditionning on that bidder 0 bids at most $\widetilde{b}$ (i.e., $V_0 + C \le \widetilde{b}$), is at most her payment $\widetilde{b}$ at the tie, therefore winning the tie does not give bidder i extra utility. This gives a necessary inequality of equilibrium bidding: $v_i + \mathbf{E}(C|V_0 + C \le \widetilde{b}) \le \widetilde{b}$ for any $v_i \in [v_l, v_r]$. Oppositely, that equilibrium must also satisfy $v_i + \mathbf{E}(C|V_0 + C \le \widetilde{b}) \ge \widetilde{b}$ for any $v_i \in [v_l, v_r]$ since otherwise the bidder would not want to bid $\widetilde{b}$ to win the auction. As a result, given that bid $\widetilde{b}$ is the equilibrium bid for all $v_i \in [v_l, v_r]$, we must have $v_i + \mathbf{E}(C|V_0 + C \le \widetilde{b}) = \widetilde{b}$ for all value $v_i \in [v_l, v_r]$.

C Proof of Theorem 2

Let $b(v)$ be any symmetric equilibrium bidding strategy (superscript * is omitted for notational convenience). Let $P(b)$, $p(b)$ be the corresponding CDF and PDF of the bid distribution. Thus, $P(b(v)) = F(v)$. Notice that Lemma 3 implies that $P(b)$ is a continuous function. Given v_i, we first compute the expected utility when bidder i bid b_i, assuming all the other uninformed bidders follow the equilibrium strategy $b(v)$. Note that the density function of the highest bid among all uninformed bidder, excluding i, is $(P^{n-1}(b))' = (n-1)P^{n-2}(b)p(b)$.

$$
\begin{aligned}
U(b_i;v_i) &= \int_{b_i\geq b, b_i\geq v_0+c} (v_i+c-\max\{v_0+c,b\})(n-1)P^{n-2}(b)p(b)f_0(v_0)g(c)db\,dv_0\,dc \\
&= \int_{b_i\geq b\geq v_0+c} (v_i+c-b)(n-1)P^{n-2}(b)p(b)f_0(v_0)g(c)db\,dv_0\,dc \\
&\quad + \int_{b_i\geq v_0+c\geq b} (v_i-v_0)(n-1)P^{n-2}(b)p(b)f_0(v_0)g(c)db\,dv_0\,dc \\
&= \int_{b_i\geq b\geq c} (v_i+c-b)(n-1)P^{n-2}(b)p(b)F_0(b-c)g(c)db\,dc \\
&\quad + \int_{b_i\geq v_0+c} (v_i-v_0)P^{n-1}(v_0+c)f_0(v_0)g(c)\,dv_0\,dc \\
&= \int_0^{b_i}\int_0^{b} (v_i+c-b)(n-1)P^{n-2}(b)p(b)F_0(b-c)g(c)dc\,db \\
&\quad + \int_0^{b_i}\int_0^{b_i-v_0} (v_i-v_0)P^{n-1}(v_0+c)f_0(v_0)g(c)\,dc\,dv_0
\end{aligned}
$$

Now, computing the derivative of $U(b_i;v_i)$ over b_i we have

$$
\begin{aligned}
U'(b_i;v_i) &= \int_0^{b_i} (v_i+c-b_i)(n-1)P^{n-2}(b_i)p(b_i)F_0(b_i-c)g(c)dc \\
&\quad + \int_0^{b_i} (v_i-v_0)P^{n-1}(b_i)f_0(v_0)g(b_i-v_0)\,dv_0 \\
&= \int_0^{b_i} (v_i-x)(n-1)P^{n-2}(b_i)p(b_i)F_0(x)g(b_i-x)dx \\
&\quad + \int_0^{b_i} (v_i-x)P^{n-1}(b_i)f_0(x)g(b_i-x)\,dx
\end{aligned}
\tag{11}
$$

where the second equality is due to the following change of variables: $(b_i-c)\to x$ in the first part; $v_0\to x$ in the second part.

Notice that, assuming $b(v)$ is the symmetric equilibrium strategy, we must have that for any v_i, $b_i=b(v_i)$ as a function of v_i makes $U'(b_i;v_i)=0$. As a result, Equation (11) must always equal 0 for $b=b(v_i)$. Using the equation $P(b_i)=F(v_i)$, we have $v_i=F^{-1}(P(b_i))$ where F^{-1} is the classical definition of the reverse function of CDF. As a result

$$
\begin{aligned}
0 &= \int_0^{b_i}(v_i-x)(n-1)p(b_i)F_0(x)g(b_i-x)dx + \int_0^{b_i}(v_i-x)P(b_i)f_0(x)g(b_i-x)\,dx \\
&= \int_0^{b_i}(F^{-1}(P(b_i))-x)(n-1)p(b_i)F_0(x)g(b_i-x)dx + \int_0^{b_i}(F^{-1}(P(b_i))-x)P(b_i)f_0(x)g(b_i-x)\,dx
\end{aligned}
\tag{12}
$$

This yield the following first-order differential equation

$$
p(b) = -\frac{P(b)\int_0^b (F^{-1}(P(b))-x)f_0(x)g(b-x)\,dx}{(n-1)\int_0^b (F^{-1}(P(b))-x)F_0(x)g(b-x)\,dx}.
$$

For the boundary condition, we derive the bid for the smallest private value l. Note that, when the private value is l with the optimal bid b, if bidder i wins, it must be the case that all the other uninformed bidders also bid b (since b is their smallest bid) and bidder 0 bids $b_0=v_0+c\leq b$. Therefore, bidder i also pays b due to ties. In this case, bidder i does not want to increase the bid to win the tie and neither wants to lower her bid to intentionally lose. This means his expected gain $l+\mathbf{E}(C|C+V_0\leq b)-b=0$. That is, $l=b-\mathbf{E}(C|C+V_0\leq b)$.

Note that such a b exists because when $b=0$, the right-hand side (RHS) is at most l where as when b is large enough the RHS will be greater than l. Since both sides are continuous, such a b exists. When there are multiple b which all satisfy $l=b-\mathbf{E}(C|C+V_0\leq b)$, the bid at l will be the largest such b. Otherwise, suppose $b'(<b)$ also satisfies $l=b'-\mathbf{E}(C|C+V_0\leq b')$ and at equilibrium all uninformed bidder's bid at l is b'. We claim that any bidder i strictly benefit by deviating to b. This is because in this case bidder i will have strictly positive utility, as opposed to utility 0 when bidding b', since her probability of

winning yet paying less than b becomes strictly positive due to other uninformed bidders bidding $b'(< b)$ at l. Therefore, at equilibrium, the bid at l must be the largest b such that $l = b - \mathbf{E}(C|C + V_0 \leq b)$.

Denote the right hand side as a function $H(b, P)$. It is easy to see that $H(b, P)$ and $\frac{\partial H(b,P)}{\partial P}$ are both continuous. Therefore, There exists a unique solution to the differential equation in a small open interval centered at the boundary value above (denoted as $\min$), and this solution can be extends to a solution over $[\min, u_c + u]$.

C.1 A Simple Numerical Algorithm

As an application of Theorem 2, it leads to a numerical algorithm for computing the equilibrium bid. From Equation 12 we have,

$$v_i = \frac{(n-1)p(b_i)\int_0^{b_i} xF_0(x)g(b_i - x)dx + P(b_i)\int_0^{b_i} xf_0(x)g(b_i - x)\,dx}{(n-1)p(b_i)\int_0^{b_i} F_0(x)g(b_i - x)dx + P(b_i)\int_0^{b_i} f_0(x)g(b_i - x)\,dx} \tag{13}$$

Before proceeding, it is worthwhile to take a look at Equation 13 for a moment. First notice that, when $n = 1$, it degenerates to the two player equilibrium in Theorem 1. Second, it is interesting that the left hand side of Equation 13 only depends on b_i and the bidding probability density $p(b)$, and importantly does not depends on v_i at all. This observation gives rise to a very clean numerical way to compute the distribution $p(b)$, assuming b_i is increasing in v_i.

Without loss of generality, assume v supports on $[0, 1]$ and c supports on $[0, 1]$. So b must supports on $[0, 2]$, as there is no incentive for a player to bid above 2. Now we divide the bid interval $[0, 2]$ into a uniform grid with $2N + 1$ points $0 = z_0 < z_1 ... < z_{2N}$ where $\Delta = \frac{1}{N}$ is the interval length between neighboring points. Let $p_i = P(z_i)$ for all $i = 0, ..., 2N$ denote the accumulated probability $\mathbb{P}(b \leq z_i)$. We wish to compute p_i.

First notice that $p_0 = 0$. This is because a bidder with valuation 0 will always bid 0 – any bid above 0 gives her some probability to win the item, in which case she will only get strictly negative utilities. Now, observe that we can approximate $p(z_i)$ by $\frac{p_i - p_{i-1}}{\Delta}$, while $P(z_i)$ simply equals p_i. Importantly, at equilibrium v_i should equal $F^{-1}(p_i)$ since we have $F(v_i) = P(b(v_i))$. Therefore, Equation 13 at $b_i = z_i$ (for all $i \geq 1$)can be approximated by

$$F^{-1}(p_{i-1}) = \frac{(n-1)\frac{p_i - p_{i-1}}{\Delta}\int_0^{z_i} xF_0(x)g(z_i - x)dx + p_i\int_0^{z_i} xf_0(x)g(z_i - x)\,dx}{(n-1)\frac{p_i - p_{i-1}}{\Delta}\int_0^{z_i} F_0(x)g(z_i - x)dx + p_i\int_0^{z_i} f_0(x)g(z_i - x)\,dx} \tag{14}$$

Therefore, if we compute p_i in a recursive manner, starting from p_1 (given that $p_0 = 0$), the only variable in the Equation 14 is p_i.